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MANIFOLDS OF J-HOLOMORPHIC DISCS

Doctoral thesis

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LJUBLJANA, 2013

UNIVERZA V LJUBLJANI
FAKULTETA ZA MATEMATIKO IN FIZIKO
ODDELEK ZA MATEMATIKO

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MNOGOTEROSTI J-HOLOMORFNIH DISKOV

Doktorska disertacija

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LJUBLJANA, 2013

Zahvala

Eni te ocenjujejo, drugi te cenijo.
Z vztrajnostjo se polni presek obojih.

Komu? Vsem, ki so verjeli in mi stali ob strani. Na prvem mestu Janji, ljubezni mojega življenja. Nato staršem in Boštjanu za podporo, Šaleškemu študentskemu oktetu, Komikazam in članom MK-ja za nora doživetja, ter Franciju, Miranu in Barbari za ponujene priložnosti.

Acknowledgements

I would like to thank Franc Forstnerič, Barbara Drinovec Drnovšek and Alex Sukhov for introducing this interesting problems to me. A major part of the work whose result is presented here was obtained during my stays at the University Lille-1, Ecole Normale Supérieure de Lyon and Institut Fourier Grenoble. I would like to thank my hosts Alex Sukhov, Jean Claude Sikorav and Hervé Gaussier for arranging the enjoyable visits. I would also like to thank Miran Černe and IMFM Ljubljana for the support provided during my doctoral study. Research on this work was supported by a fellowship from the Agency for Research and Development (ARRS) of the Republic of Slovenia.

Abstract

We study the deformation space of J-holomorphic discs in an almost complex manifold (M, J). We find a nonlinear invertible operator mapping the space of all small J-holomorphic deformations of the given J-holomorphic disc onto the space of small holomorphic deformations of the standard disc in $\mathbb{C}^n$. Furthermore, we study the neighborhood of an embedded disc attached along its boundary to a maximal totally real submanifold. In particular, we provide sufficient conditions for the set of all nearby attached J-holomorphic discs to be a manifold. Finally, we extend the theorem of E. Poletsky, concerning plurisubharmonicity of the envelope of the Poisson functional of an upper semicontinuous function, to almost complex surfaces of real dimension 4.

Math. Subj. Class. (MSC 2000): 32Q60,32Q65,35JXX,32U05

Keywords: almost complex manifolds, J-holomorphic discs, Pascali systems, plurisubharmonic functions.

Povzetek

Disertacija proučuje deformacije J-holomorfnih diskov na skoraj kompleksnih mnogoterostih. Vzpostavljena je povratno enolična zveza med malimi J-holomorfnimi perturbacijami danega J-holomorfnega diska in malimi holomorfnimi deformacijami standardnih diskov v $\mathbb{C}^n$. Nadalje so podani zadostni pogoji, pod katerimi ima množica bližnjih diskov pripetih na maksimalne totalno realne mnogoterosti strukturo mnogoterosti. Delo se zaključi s posplošitvijo rezultata E. Poletskyija, ki govori o plurisubharmoničnosti ogrinjače Poissonovega funkcionala navzgor polzvezne funkcije. Dokazan je oblika trditve za skoraj kompleksne mnogoterosti realne dimenzije 4.

Math. Subj. Class. (MSC 2000): 32Q60,32Q65,35JXX,32U05

Ključne besede: skoraj kompleksne mnogoterosti, J-holomorfni diski, Pascalijevi sistemi, plurisubharmonične funkcije.

Contents

Introduction

The main object of the dissertation are smooth manifolds M of even dimension, equipped with a non-integrable almost complex structure, called almost complex manifolds. An almost complex structure J is a real endomorphism defined on the tangent bundle TM with property $J^2 = -Id$. In general, there are no holomorphic functions or maps on such manifolds, but there are plenty of pseudoholomorphic curves.

The study of this subject began in the 1960's, but has revolutionized since the paper of Mikhail Gromov [**15**] was published in 1985. He proved, by means of almost complex analysis, the existence of global symplectic invariants. These techniques have found further applications in topology and other fields. Pseudoholomorphic curves are used to define Floer homology and to study (non-)equivalence of smooth structures.

In this work we study the theory of J-holomorphic discs, that is, maps $u\colon \mathbb{D} \to (M, J)$. We denote by $\mathbb{D}$ the unit disc in $\mathbb{C} \cong (\mathbb{R}^2, J_{st})$. The local existence of J-holomorphic discs goes back to the work of Nijenhuis and Woolf [**28**]. They presented both an analytic and a geometric approach to the $\bar{\partial}_J$-equation satisfied by J-holomorphic discs:

$$\bar{\partial}_J u = \frac{1}{2}\left(\mathrm{d}u + J \circ \mathrm{d}u \circ J_{st}\right) = 0.$$

The theory has been pushed forward in many directions and it now allows to generalize some of the results and applications from the standard complex theory, especially those obtained by methods using analytic discs. Furthermore, some of the results that were obtained by these methods are new even in the integrable case.

The dissertation is organized as follows. After introducing almost complex manifolds in § 1.1, we present a local form of the above $\bar{\partial}_J$-equation in § 1.2:

$$u_{\bar{\zeta}} + A(u)\overline{u_{\zeta}} = 0, \quad \zeta = x + iy \in \mathbb{D},$$

valid for $u\colon \mathbb{D} \to \mathbb{C}^n$. We call $A := A(J)$ the complex matrix of J.

The complex matrix A vanishes if and only if J is the standard structure on $\mathbb{R}^{2n}$. In general, it depends on the local chart. We give a rule for transformation of A under a change of coordinates and present the integrability conditions for the almost complex structure in terms of the complex matrix.

Further, we study a neighborhood of a given point $p \in M$ in § 1.3. We show that one can always fix a chart φ around p so that $\varphi(p) = 0$ and that the direct image $\varphi_*(J)$ coincides with the standard structure at the origin. Furthermore, shrinking the neighborhood, the structure can be viewed as a small second order deformation of the standard one on the whole neighborhood. That means that the $\mathcal{C}^1$-norm of the complex matrix A is small. The Diedrich-Sukhov normalization theorem [**9**] states that the complex derivatives of A vanish at the origin when we choose an appropriate diffeomorphism. In contrast, vanishing of the anti-complex derivatives is related to the integrability of the almost complex structure.

Similarly, one can consider a structure that is standard along the image of a given flat J-holomorphic disc $u_0(\mathbb{D}) = \{0\}^{n-1} \times \mathbb{D} \subset \mathbb{C}^n$. In § 1.4 we show that the general case of an embedded J-holomorphic disc can be reduced to this simple case [**19**]. Further, we obtain vanishing of the complex derivatives of A along $u_0(\mathbb{D})$ by the Sukhov-Tumanov normalization [**37**]. Due to simplicity we state the result for $\mathbb{C}^2$. However, it remains valid also in higher dimensions [**33**].

We conclude the first chapter with an original result for $(\mathbb{R}^4, J)$. We show that, unlike in the neighborhood of a point, the structure is not standard up to the second order on a neighborhood of $u_0(\mathbb{D})$, that is, A has to be approximated by a non-integrable complex matrix. We prove that by a change of coordinates preserving the flat disc $u_0(\mathbb{D})$ and the property $J = J_{st}$ along its image we can make J as close as desired to a model almost complex structure J_β of the form

$$J_\beta(z,w) = \begin{bmatrix} J_{st} & \begin{matrix} -2\Im(z\beta(w)) & -2\Re(\bar{z}\beta(w)) \\ -2\Re(\bar{z}\beta(w)) & 2\Im(z\beta(w)) \end{matrix} \\ 0 & J_{st} \end{bmatrix}$$

(see Theorem 1.25). The class of all model structures is classified by means of a complex function whose $\beta\colon \mathbb{D} \to \mathbb{C}$ whose modulus is a biholomorphic invariant [**22**]. Our reduction to such normal form is similar to the Sukhov-Tumanov normalization.

The main goal of the second chapter is to present some facts from the classical theory of elliptic equations. Although this chapter does not contain any original results we present a coherent survey of the theory that has so far been scattered all over the literature. We hope that this could serve as the basis for a survey-expository paper on this subject. The linear theory is relevant to our dissertation since such systems appear when we linearize the J-holomorphicity condition.

We introduce the Cauchy-Green operator

$$T(u)(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{u(\zeta)}{z-\zeta}\,\mathrm{d}x\,\mathrm{d}y(\zeta).$$

that solves the usual $\bar{\partial}$-equation. We study general analytic functions, that is, scalar complex functions solving the equation

$$u_{\bar{\zeta}} + B_1 u + B_2 \bar{u} = 0,$$

and their higher dimensional analogue. It is pointed out how the scalar theory differs from the higher dimensional one. When $n = 1$, the following Fredholm operator mapping the space $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$ to itself,

$$\Phi(u) = u + T(B_1 u + B_2 \bar{u}) = \phi,$$

gives a bijective correspondence between the generalized and the usual analytic functions. In contrast, when $n \geq 2$, its kernel might not be trivial. We include here a recent result of Sukhov and Tumanov [**36**] which shows how to modify this operator to obtain a one-to-one correspondence in general. In conclusion we study the Riemann-Hilbert problem for Pascali systems since such type of boundary conditions arises when we attach the discs to a totally real submanifold.

The last chapter is the most intensive in the sense of producing new results in the theory of J-holomorphic discs. First, in § 3.1 we write the local system in an equivalent form

$$\Phi^J(u) := u + TA(u)\overline{u_\zeta} = 0, \quad \zeta = x + iy \in \mathbb{D}.$$

The operator Φ^J provides a correspondence between the J-holomorphic and the usual holomorphic discs of the same regularity. In a neighborhood of a given point, such a correspondence is only a small perturbation of the identity. We use implicit function theorem to prove existence of a small disc through any point and in any tangent direction. This is the classical Nijenhuis-Woolf theorem. We present here a slightly improved version from [**19**] interpolating also the higher order jets.

The problem changes when we deal with big J-holomorphic discs. Locally the above correspondence can be viewed as a compact (not small) perturbation of the identity with a Fredholm derivative similar to the operator Φ above. The case when $n = 2$ is special since there is only one normal direction. One can apply the theory of generalized analytic functions in case of an embedded disc. We prove invertibility by using the approximation with the model structure [**22**]. This particular result was recently strengthened for higher dimensions in [**36**]. It was shown that the set of pseudoholomorphic discs form a smooth Banach manifold modelled on the standard holomorphic discs in $\mathbb{C}^n$ of the same regularity. We include their theorem.

A similar approach is used to study the neighborhood of a disc attached to maximal totally real submanifold in § 3.2. The local system turns into a boundary value problem. We use the implicit function theorem to describe a sufficent condition for the set $\mathcal{M}$ of nearby attached pseudoholomorphic discs to be a manifold.

When dealing with an integrable structure and dimension $n = 2$, such a condition was first given in terms of the Maslov index. This index describes the relation between the initial disc and the boundary submanifold [**12**]. It was proved later that the same result remains valid in the non-integrable case [**18**]. We give here a new interpretation of this result using the above mentioned connection between the generalized analytic functions and the usual holomorphic functions.

In higher dimensions, $n \geq 3$, the integrable case was studied by Globevnik [**14**]. The set $\mathcal{M}$ is known to depend on partial indices provided by the Birkhoff matrix factorization. In particular, it is a manifold if they are all greater or equal to -1. We extend this result to the almost complex case [**21**]. It is pointed out how the problem depends not only on the partial indices but also on the almost complex structure J (this is not the case when $n = 2$!). In certain special cases Globevnik's sufficient condition remains the same, but in general the relation

$$\sum_{\kappa_j < -1} (-\kappa_j - 1) + N - r = 0$$

has to be fulfilled. Here N is the kernel dimension of a certain integral transformation based on the Cauchy-Green transformation and r is the solvability defect for the associated integral equation introduced in the Ph. D. thesis of Buchanan [**4**].

In § 3.3 and § 3.4 we apply the developed theory. Firstly, we give a holomorphic approximation of a disc with small $\overline{\partial}$-derivative in the space $\mathbb{R}^{2n}$. The result is new and might be the initial step in giving a complete answer to a similar question in the general non-integrable case [**23**]. Further, we present a method from [**7**] for gluing pseudo-holomorphic discs to a certain real torus. Such a construction is one of the two crucial ingredients needed in the last section where we prove the existence of a maximal plurisubharmonic minorant of a given upper semicontinous function f on an almost complex manifold (M, J) of complex dimension two. The maximal function is obtained as the pointwise minimum of averages of f over the boundaries of all J-complex discs in M centered at the given point. This is an almost complex analogue of classical results of Poletsky [**31**], Lárusson and Sigurdsson [**24**], Rosay [**32, 34**] and others in the case when the almost complex structure J is integrable. The result was recently published in [**23**]. We state it also in the theorem below.

Theorem. *Suppose that (M, J) is a smooth almost complex manifold with $J \in \mathcal{C}^\infty$ and $\dim_{\mathbb{R}} M = 4$. Given an upper semicontinuous function $f\colon M \to \mathbb{R} \cup \{-\infty\}$ and a point $p \in M$, define*

$$\hat{f}(p) = \inf \int_0^{2\pi} f \circ u(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi}.$$

The infimum is over all J-holomorphic discs $u\colon \overline{\mathbb{D}} \to M$ with $u(0) = p$. Then the function $\hat{f}$ is J-plurisubharmonic or identically $-\infty$.

CHAPTER 1

Local notion of J-holomorphicity

1.1. Almost complex manifolds

Let V be a real vector space and I the identity map. An endomorphism $J\colon V \to V$ is called *complex structure* if $J^2 = -I$.

Example 1.1. The first and the most natural example of a complex structure is the *standard structure* on $\mathbb{R}^{2n}$. We represent it in the standard coordinate base $(x_1, y_1, \dots, x_n, y_n)$ by a $(2n \times 2n)$ matrix

$$J_{st}^{(2n)} = \begin{bmatrix} 0 & -1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{bmatrix}.$$

We drop the upper index since the dimension will always be clear from the context. This endomorphism corresponds to the multiplication by i if we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ equipped with the usual coordinates $(z_1, z_2, \dots, z_n)$, $z_j = x_j + iy_j$.

Example 1.2. Let J be a complex structure defined on $\mathbb{R}^2$. Its matrix representation is always of the form

$$\begin{bmatrix} c & -d \\ \frac{c^2+1}{d} & -c \end{bmatrix},$$

where $c \in \mathbb{R}$ and $d \in \mathbb{R}\setminus\{0\}$.

Remark 1.3. Suppose $v_1 \in V$ is a nonzero vector. It is obvious that for a complex structure J the vectors v_1 and Jv_1 are linearly independent. If we choose $v_2 \notin \operatorname{Span}\{v_1, Jv_1\}$, $v_3 \notin \operatorname{Span}\{v_1, Jv_1, v_2, Jv_2\}$ and so forth, we see that admitting an almost complex structure, the vector space has to be of even dimension. Furthermore, applying the constructed basis J is represented by the matrix J_{st}.

Every even-dimensional space admits a complex structure. Indeed, we can define for instance $Je_{2j-1} := e_{2j}$ and $Je_{2j} := -e_{2j-1}$ for some vector base e_j, $j = 1, 2, \ldots, 2n$. Furthermore, one can define multiplication by complex numbers as

$$(\lambda + i\mu) \cdot v \mapsto \lambda v + \mu J v.$$

Thus V becomes a complex vector space whose complex dimension equals one half of the real one.

Our aim is to study smooth manifolds whose tangent spaces are at every point equipped with a complex structure. We will mostly follow [**25**] when introducing this objects. However, before giving any exact definitions, let us motivate the discussion by showing that such a requirement is fulfilled in the case of complex manifolds.

Let M be a smooth manifold of real dimension $2n$. Let $\{\varphi_\eta\}_\eta$ be a set of compatible holomorphic charts mapping open subsets U_η of M to the space $\mathbb{C}^n \simeq (\mathbb{R}^{2n}, J_{st})$. For $p \in U_\eta$ the tangent space T_pM is the set of equivalence classes $[\eta, v]$ with $v \in \mathbb{R}^{2n}$ under the relation

$$[\eta, v] \sim [\eta', v'] \Longleftrightarrow v' = \mathrm{d}(\varphi_{\eta'} \circ \varphi_\eta^{-1})(x)v,$$

where $x = \varphi_\eta(p) \in \mathbb{R}^{2n}$. We define $J_p \colon T_pM \to T_pM$ as

$$J_p\left([\eta, v]\right) = [\eta, J_{st}v]\,.$$

When $U_\eta \cap U_{\eta'} \neq \emptyset$ the transition map $\varphi_{\eta'} \circ \varphi_\eta^{-1}$ is holomorphic. This means that its differential commutes with J_{st}. Thus J_p is well defined. Patching such structures together, we obtain a tensor field defined on the whole tangent bundle TM.

A natural question now arises. Let M be only a smooth manifold. Suppose that there exists a tensor field on TM applying to every tangent space a linear complex structure. Does this imply the existence of a complex structure on M? The answer is in general negative!

Definition 1.4. *Let M be a $\mathcal{C}^\infty$-smooth real manifold of real dimension $2n$. An endomorphism $J \in \mathrm{End}_{\mathbb{R}}(TM)$ is an almost complex structure if the fibers are complex structures, that is, $J_p^2 = -I$ for every $p \in M$. The pair (M, J) is called an almost complex manifold.*

Example 1.5. Let U be an open set in $\mathbb{R}^{2n}$ and let $J \colon U \to \mathbb{R}^{2n \times 2n}$ be a matrix-valued function such that $J(p)^2 = -I$ for every $p \in \mathbb{R}^{2n}$. Then the pair (U, J) is an almost complex manifold. In particular, if $J \equiv J_{st}$ then (U, J) can be viewed as a complex submanifold of $\mathbb{C}^n$.

Remark 1.6. Note that one can always work locally in $\mathbb{R}^{2n}$. Suppose that $\varphi\colon U \to \mathbb{R}^{2n}$ is a smooth chart on M. Then $\varphi(U)$ is an almost complex manifold equipped with the direct image structure

$$\varphi_*(J) = \mathrm{d}\varphi \circ J \circ \mathrm{d}\varphi^{-1}.$$

Further, given $p \in M$ one can make a linear change of coordinates such that $\varphi(p) = 0$ and that $\varphi_*(J)(0) = J_{st}$ (see Remark 1.3).

Remark 1.7. We will often consider almost complex structures J and manifolds of lower regularity. We stress that $\mathcal{C}^k$ regularity of J is preserved by $\mathcal{C}^{k+1}$-local diffeomorphisms.

Example 1.8. Every oriented hypersurface $\Sigma \subset \mathbb{R}^3$ carries an almost complex structure which it inherits from the usual vector product $\times$. Let ν be the Gauss map which associates to every point $p \in \Sigma$ the outward unit normal vector ν_p. The almost complex structure on Σ is then given by the formula

$$J_p u = \nu_p \times u.$$

Example 1.9. The vector space $\mathbb{R}^7$ also carries a vector product (we again denote it by $\times$). It is bilinear and skew symmetric, and related to the standard inner product $\langle\, .\, ,\, .\, \rangle$ by the rules:

$$\langle u \times v, w \rangle = \langle u, v \times w \rangle\,,$$

$$(u \times v) \times w + u \times (v \times w) = 2\,\langle u, w \rangle\, v - \langle v, w \rangle\, u - \langle v, u \rangle\, w.$$

This product arises by viewing $\mathbb{R}^8$ as the algebra of Cayley numbers (we view $\mathbb{R}^7$ as the set of imaginary Cayley numbers and $u \times v$ as the imaginary part of the product of u and v in $\mathbb{R}^8$). It follows as in the previous example that every oriented hypersurface $\Sigma \subset \mathbb{R}^7$ carries an almost complex structure J defined ν by $J_p u = \nu_p \times u$, where ν is again the Gauss map.

As seen above, whenever the manifold is of even dimension one can define an almost complex structure locally. However, there exist manifolds without global almost complex structure. For instance, S^2 and S^6 are the only spheres that admit such a structure. There always exists an almost complex structure coming from the complex one, we call it *integrable*. But, as pointed out earlier not every structure is integrable. For example, the above mentioned almost complex structure on S^6 was shown to be non-integrable and it is not currently known whether or not the sphere S^6 has a complex structure (see also [**25**]).

A necessary and sufficient condition for an almost complex structure J on M to be integrable is the existence of an atlas $\{\varphi_\eta, U_\eta\}$ on M such that on every chart the transported structure $(\varphi_\eta)_*(J)$ is standard. Hence for $U_\eta \cap U_{\eta'} \neq \emptyset$ the differential of $\varphi_{\eta'} \circ \varphi_\eta^{-1}$ commutes with J_{st} and the transition maps are holomorphic.

On a real surface every almost complex structure is integrable. Let (x, y) be the standard coordinates in $\mathbb{R}^2$. It suffices to find a smooth diffeomorphism φ mapping a neighborhood of the unit disc $\mathbb{D}$ to some open subset of $(\mathbb{R}, J_{st}) \simeq \mathbb{C}$ such that

$$J_{st} = \mathrm{d}\varphi \circ J \circ \mathrm{d}\varphi^{-1} \Longleftrightarrow i\mathrm{d}\varphi = \mathrm{d}\varphi \circ J.$$

In addition, we may assume that $J(0) = J_{st}$ (Remark 1.6). Hence by Example 1.3 there exist real smooth functions $c(x, y)$ and $d(x, y) > 0$ such that $c(0) = 0$, $d(0) = 1$ and

$$J\left(\frac{\partial}{\partial y}\right) = -d(x, y)\frac{\partial}{\partial x} - c(x, y)\frac{\partial}{\partial y}.$$

For $\zeta = x + iy \in \mathbb{C}$ we have the usual relations

$$\frac{\partial \varphi}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial \varphi}{\partial x} - i\frac{\partial \varphi}{\partial y}\right), \quad \frac{\partial \varphi}{\partial \bar\zeta} = \frac{1}{2}\left(\frac{\partial \varphi}{\partial x} + i\frac{\partial \varphi}{\partial y}\right),$$
$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial \zeta} + \frac{\partial \varphi}{\partial \bar\zeta}, \qquad \frac{\partial \varphi}{\partial y} = i\left(\frac{\partial \varphi}{\partial \zeta} - \frac{\partial \varphi}{\partial \bar\zeta}\right).$$

Thus the above question is equivalent to solving the equation

$$\frac{\partial \varphi}{\partial \bar\zeta} = \frac{1 - d(\zeta, \bar\zeta) - ic(\zeta, \bar\zeta)}{1 + d(\zeta, \bar\zeta) - ic(\zeta, \bar\zeta)}\frac{\partial \varphi}{\partial \zeta} = \mu(\zeta, \bar\zeta)\frac{\partial \varphi}{\partial \zeta}.$$

Observe that $|\mu| < q < 1$ on $\mathbb{D}$. Hence this is a classical Beltrami equation whose solutions we discuss in § 2.2.

A smooth map f from an almost complex manifold (M', J') to (M, J) is *(J',J)-holomorphic* if its differential satisfies

$$\bar\partial_{J',J} f = \frac{1}{2}\left(\mathrm{d}f + J \circ \mathrm{d}f \circ J'\right) = 0 \ \text{ on } TM'.$$

Generically, if $\dim_{\mathbb{R}} M' > 2$ the system is overdetermined and there are no holomorphic maps. The special case is when $\dim_{\mathbb{R}} M' = 2$. The structure J' is then integrable and (M', J') can be locally viewed as the unit disc $\mathbb{D} \subset \mathbb{R}^2$ equipped with J_{st}, or simply as the unit disc $\mathbb{D} \subset \mathbb{C}$. We call such f a *pseudoholomorphic curve* (or *J-holomorphic curve* respectively). Their existence was proved by Nijenhuis and Woolf [**28**]. In this work we study the particular case when $(M', J') = (\mathbb{D}, J_{st})$.

1.2. Complex matrix of an almost complex structure

Let (M, J) be an almost complex manifold. By a *disc in* M we mean a $\mathcal{C}^1$-map from a neighborhood of the closed unit disc $\overline{\mathbb{D}} \subset \mathbb{C}$ to M. The disc u is said to be *J-holomorphic* if its differential satisfies

$$\bar{\partial}_J(u) = \frac{1}{2}\left(\mathrm{d}u + J(u) \circ \mathrm{d}u \circ J_{st}\right) = 0. \tag{1.1}$$

This corresponds to the (J_{st}, J)-holomorphicity introduced above.

We denote by $\zeta = x + iy$ the complex variable on $\mathbb{C}$. Locally, when $M = \mathbb{R}^{2n}$ the above system is equivalent to

$$\frac{\partial u}{\partial y} = J(u)\frac{\partial u}{\partial x}. \tag{1.2}$$

We assume throughout this chapter that J and u are of class $\mathcal{C}^1$ at least. Note that to preserve this smoothness the diffeomorphisms has to be of class $\mathcal{C}^2$ (Remark 1.6).

1.2.1. The complex linear operator. The next step is to write (1.4) in the complex notation. We follow [**33**] and [**38**]. Since we are working in the real space $\mathbb{R}^{2n}$ we have to change the notation when we define the relations:

$$\frac{\partial u}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - J_{st}\frac{\partial u}{\partial y}\right), \quad \frac{\partial u}{\partial \bar{\zeta}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + J_{st}\frac{\partial u}{\partial y}\right),$$
$$\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial \zeta} + \frac{\partial u}{\partial \bar{\zeta}}\right), \qquad \frac{\partial u}{\partial y} = J_{st}\left(\frac{\partial u}{\partial \zeta} - \frac{\partial u}{\partial \bar{\zeta}}\right).$$

Restricting ourselves to the case when $\det(J(u) + J_{st}) \neq 0$ (this in particular happens when $J(u)$ is close to J_{st}) one has the following equation for J-holomorphic discs:

$$\frac{\partial u}{\partial \bar{\zeta}} + Q(u)\frac{\partial u}{\partial \zeta} = 0, \tag{1.3}$$

where $Q(u) := (J(u) + J_{st})^{-1}(J(u) - J_{st})$.

Lemma 1.10. *Let $J\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a $\mathbb{R}$-linear map such that $J + J_{st}$ is invertible. Set $Q = (J + J_{st})^{-1}(J - J_{st})$. Then $J^2 = -I$ is equivalent to Q being conjugate linear, that is, $QJ_{st} + J_{st}Q = 0$.*

Proof. If $J^2 = J_{st}^2 = -I$ we have

$$J_{st}(J + J_{st})^{-1} = (J + J_{st})^{-1}J \quad \text{and} \quad J(J - J_{st}) = -(J - J_{st})J_{st}.$$

Therefore we have

$$J_{st}Q = J_{st}(J + J_{st})^{-1}(J - J_{st}) = -(J + J_{st})^{-1}(J - J_{st})J_{st} = -QJ_{st}.$$

On the other hand, if a conjugate linear operator Q is given one can express J from $(J + J_{st})Q = (J - J_{st})$. It equals to

$$J = J_{st}(I + Q)(I - Q)^{-1}.$$

Note that $I - Q = 2(J + J_{st})J$. Hence, $I - Q$ it is invertible whenever $J + J_{st}$ is invertible. Further, note that the conjugate linearity of Q implies $J_{st}(I + Q) = (I - Q)J_{st}$ and $(I - Q)^{-1}J_{st}(I + Q) = J_{st}$. Thus one has

$$\begin{aligned} J^2 &= J_{st}(I + Q)(I - Q)^{-1}J_{st}(I + Q)(I - Q)^{-1} \\ &= J_{st}(I + Q)J_{st}(I - Q)^{-1} = (I - Q)J_{st}^2(I - Q)^{-1} = -I. \end{aligned}$$

□

Remark 1.11. The conjugate linearity of Q implies a natural property of pseudoholomorphic discs. If the map $\zeta \mapsto u(\zeta)$ is J-holomorphic, so is $\zeta \mapsto u(\lambda\zeta)$, for any fixed complex number $|\lambda| < 1$.

The operator Q has a long history; in particular, for integrable complex structures, it was called the deformation tensor. However, the equation (1.3) still refers to the space $\mathbb{R}^{2n}$. It is our desire to work in complex spaces. Hence we would like to replace Q by a complex linear operator. We obtain it by conjugating a given vector $v \in \mathbb{R}^{2n}$ before mapping it with Q. Here of course $\bar{v} \in \mathbb{R}^{2n}$ is defined in a natural way so that it corresponds to the complex analog.

Proposition 1.12. *Let us introduce the spaces*

$$\mathcal{J} = \left\{J \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n};\ J \text{ is } \mathbb{R}\text{−linear},\ J^2 = -I, \det(J_{st} + J) \neq 0\right\},$$

$$\mathcal{A} = \left\{A \colon \mathbb{C}^{n} \to \mathbb{C}^{n};\ A \text{ is } \mathbb{C}\text{−linear},\ \det(I - A\bar{A}) \neq 0\right\}.$$

Let Q be defined as in Lemma 1.10 for $J \in \mathcal{J}$. Then there exists a unique matrix $A \in \mathcal{A}$, such that $Av = Q\bar{v}$ for every $v \in \mathbb{C}^n \simeq \mathbb{R}^{2n}$. Furthermore, in J can be expressed in terms of A as

$$Jv = J_{st}(I - A\bar{A})^{-1}\left((I + A\bar{A})v + 2A\bar{v}\right).$$

Proof. If Q is conjugate linear, then 1 is an eigenvalue of Q if and only if -1 is an eigenvalue of Q. This in turn holds if and only if 1 is an eigenvalue of Q^2. But note that $Q^2 = A\bar{A}$. Hence Q has eigenvalues ± 1 if and only if $A\bar{A}$ has an eigenvalue 1.

Let $J \in \mathcal{J}$. We claim that the condition $\det(J_{st} + J) \neq 0$ along with $J^2 = -I$ implies that -1 is not an eigenvalue of Q. Indeed, assume there is $v \in \mathbb{R}^{2n}$ such that $Qv = (J + J_{st})^{-1}(J - J_{st})v = -v$. Obviously one has $Jv = 0$ and thus $v = 0$. Hence by the above consideration 1 is not an eigenvalue of $A\bar{A}$, that is, $A \in \mathcal{A}$.

Conversely, given $A \in \mathcal{A}$, we show that there exists a unique $J \in \mathcal{J}$, such that $J \mapsto A$. We define a conjugate linear operator Q as $Qv = A\bar{v}$. Since $A\bar{A}$ does not have 1 as an eigenvalue, the same is true for Q. As in the proof of Lemma 1.10 one can define $J = J_{st}(I+Q)(I-Q)^{-1} \in \mathcal{J}$.

Finally, let us state J in terms of A. Note that

$$(I + Q)(I - Q) = (I - Q)(I + Q) = I - Q^2 = I - A\bar{A}.$$

Thus the terms $(I + Q)$ and $(I - Q)^{-1}$ commute. Hence the above definition of J can be turned into

$$\begin{aligned} J &= J_{st}(I - Q)^{-1}(I + Q) = J_{st}(I - A\bar{A})^{-1}(I + Q)^2 \\ &= J_{st}(I - A\bar{A})^{-1}(I + 2Q + A\bar{A}). \end{aligned}$$

This completes the proof. □

Let us assume now that the structure $J(u) + J_{st}$ is invertible along the image of a mapping $u\colon \overline{\mathbb{D}} \to \mathbb{R}^{2n}$. The J-holomorphicity condition (1.3) can be written as:

$$\frac{\partial u}{\partial \bar{\zeta}} + A(u)\overline{\frac{\partial u}{\partial \zeta}} = 0, \tag{1.4}$$

where $A(Z)(v) = (J_{st} + J(Z))^{-1}(J(Z) - J_{st})(\bar{v})$ is a complex linear endomorphism for every $Z \in \mathbb{C}^n$. We call it the *complex matrix of the endomorphism* J. The entries of $A(Z)$ are functions of class the same regularity as J in Z. Furthermore, the condition $A(Z) = 0$ is equivalent to $J(Z)$ being equal to J_{st}.

The advantage of this notion is that one can consider A as a $n \times n$ matrix with complex coefficients and u as a complex vector function mapping from a neighborhood of $\overline{\mathbb{D}}$ into $\mathbb{C}^n$. Thus, when working locally, one can consider J-holomorphic discs in $\mathbb{C}^n$ satisfying (1.4).

1.2.2. Transformation rule. The definition of the complex matrix A is local, that is, it depends on the coordinates we have chosen (to begin with $J(u) + J_{st}$ must be invertible). It is our desire to explain in the sequel which simple forms of A can be obtained by choosing an appropriate diffeomorphism (we will call them *normalizations*). In order to do so we will introduce a transformation rule for complex matrices under diffeomorphisms from [**37**] (Lemma 1.15).

We begin by stating the Nijenhuis-Woolf theorem [**28**] that was already mentioned in the first section. We shall need it twice for what follows. However, we postpone its proof until § 3.1 (Theorem 3.1).

THEOREM 1.13 (Nijenhuis-Woolf theorem). *Let M be an almost complex manifold equipped with a $\mathcal{C}^1$-smooth structure J. For every $p \in M$ and $v \in T_pM$ there exist $\lambda > 0$ and a J-holomorphic disc $u_{p,v}$ such that $u_{p,v}(0) = p$ and $\mathrm{d}u_{p,v}(0)\left(\frac{\partial}{\partial x}\right) = \lambda v$. If $J \in \mathcal{C}^\infty$ the disc $u_{p,v}$ depends smoothly on the data (p, v).*

Remark 1.14. Note that instead of fixing the direction $\mathrm{d}u_{p,v}(0)\left(\frac{\partial}{\partial x}\right)$ one can fix the direction of

$$\mathrm{d}u_{p,v}(0)\left(\frac{\partial}{\partial \zeta}\right) = \mathrm{d}u_{p,v}(0)\left(\frac{\partial}{\partial x} - J_{st}\frac{\partial}{\partial y}\right) = \lambda(v + Jv),$$

since for $w \in T_pM$ one has $w = 1/2\left[(w - Jw) + J(w - Jw)\right]$.

Let φ be a local diffeomorphism between two open subsets of $\mathbb{C}^n$. Here and throughout this chapter we denote by $Z = (z_1, z_2, \ldots, z_n)$ the complex coordinates at the source and by $Z' = (z_1', z_2', \ldots, z_n')$ the new coordinates. For $p \in \mathbb{C}^n$ we denote by $\varphi_Z(p)$, $\varphi_{\bar Z}(p)$ the matrices

$$(1.5)\ \varphi_Z(p) = \left(\frac{\partial z_j'}{\partial z_k}(p)\right)_{j,k=1}^n \quad \text{and} \quad \varphi_{\bar Z}(p) = \left(\frac{\partial z_j'}{\partial \bar z_k}(p)\right)_{j,k=1}^n.$$

Let $\tau \in \left\{\zeta, \bar\zeta\right\}$. Given a disc u the usual chain rule equals to

$$\left[\varphi(u)\right]_\tau = \varphi_Z(u)u_\tau + \varphi_{\bar Z}(u)(p)\overline{u}_\tau.$$

We assume the diffeomorphisms to be defined on a neighborhood of a given point or on a neighborhood of $u(\overline{\mathbb{D}})$, where u is the disc under our consideration. We omit stating this every time. In addition we say that φ *fixes a point* $p \in \mathbb{C}^n$ if $\varphi(p) = p$. Furthermore, it *fixes a disc* if it fixes every point of its image.

LEMMA 1.15. *Let J be an almost complex structure defined on an open set $U \subset \mathbb{R}^{2n}$ such that $\det(J + J_{st}) \neq 0$. We define the matrix A as in (1.4). Let u be a J-holomorphic disc in U and φ a diffeomorphism such that $(\overline{\varphi_Z} + \overline{\varphi_{\bar Z}}A) \neq 0$ on U. Then for A and its direct image $A' = \varphi_*(A)$ the following holds:*

$$A' \circ \varphi = \left(\varphi_Z A - \varphi_{\bar Z}\right)\left(\overline{\varphi_Z} - \overline{\varphi_{\bar Z}}A\right)^{-1} \text{ on } u(\overline{\mathbb{D}}).$$

PROOF. We will write $\varphi_Z = \varphi_Z(u)$ and $\varphi_{\bar{Z}} = \varphi_{\bar{Z}}(u)$. First we apply the chain rule and J-holomorphicity of u to get

$$[\varphi(u)]_{\bar{\zeta}} = \left(\varphi_Z u_{\bar{\zeta}} + \varphi_{\bar{Z}}\bar{u}_{\bar{\zeta}}\right) = \left(-\varphi_Z A(u) + \varphi_{\bar{Z}}\right)\overline{u_\zeta}.$$

On the other hand, one has $[\varphi(u)]_{\bar{\zeta}} + A'(\varphi(u))\overline{[\varphi(u)]_\zeta} = 0$. Thus

$$[\varphi(u)]_{\bar{\zeta}} = -A'(\varphi(u))\left(\overline{\varphi_Z u_\zeta} + \overline{\varphi_{\bar{Z}}\bar{u}_\zeta}\right) = -A'(\varphi(u))\left(\overline{\varphi_Z} - \overline{\varphi_{\bar{Z}}}A(u)\right)\overline{u_\zeta}.$$

Thus for every point ζ in $\overline{\mathbb{D}}$ the following holds:

$$\left[A'\left(\varphi(u)\right)\left(\overline{\varphi_Z} - \overline{\varphi_{\bar{Z}}}A(u)\right) - \left(\varphi_Z A(u) - \varphi_{\bar{Z}}\right)\right]\overline{u_\zeta} = 0.$$

But note that by Nijenhuis-Woolf theorem there exists a J-holomorphic disc centered at any point, with a prescribed direction of derivative. Thus $\overline{u_\zeta}$ can be replaced by an arbitrary $v \in \mathbb{C}^n$. □

1.2.3. Integrability conditions. Our next goal is to express the integrability conditions of a given almost complex structure in terms of the complex matrix A. We follow [**36**]. Let f be a function mapping from an open set $U \subset (\mathbb{R}^{2n}, J)$ to $\mathbb{C}$. We denote by f_Z and $f_{\bar{Z}}$ the row vectors with partial derivatives of f as entries

$$f_Z(p) = \left(\frac{\partial f}{\partial z_j}\right)_{j=1}^n, \quad f_{\bar{Z}}(p) = \left(\frac{\partial f}{\partial \bar{z}_j}\right)_{j=1}^n.$$

Again, if u is a disc, for $\tau \in \left\{\zeta, \bar{\zeta}\right\}$ the usual chain rule gives

$$[f \circ u]_\tau = f_Z(u)u_\tau + f_{\bar{Z}}(u)\bar{u}_\tau.$$

We claim that f is (J, J_{st})-holomorphic if and only if

$$f_{\bar{Z}} - f_Z A(f) = 0. \tag{1.6}$$

Indeed, let u be a J-holomorphic disc. We have

$$[f \circ u]_{\bar{\zeta}} = f_Z(u)u_{\bar{\zeta}} + f_{\bar{Z}}(u)\bar{u}_{\bar{\zeta}} = \left(f_{\bar{Z}}(u) - f_Z(u)A(u)\right)\overline{u_\zeta}.$$

Assume f is (J, J_{st})-holomorphic. Then $[f \circ u]_{\bar{\zeta}}$ vanishes. By Nijenhuis-Woolf theorem there exists a J-holomorphic disc centered at in a given point and with a prescribed direction of u_ζ. Hence (1.6) holds.

Conversely, let u be a J-holomorphic disc centered at $p \in \mathbb{R}^{2n}$, such that for $v \in T_p\mathbb{R}^{2n}$ we have $\mathrm{d}u(0)\left(\frac{\partial}{\partial x}\right) = \lambda v$. Then if (1.6) holds, $f \circ u$ is J_{st}-holomorphic. Hence we have

$$J_{st} \circ \mathrm{d}f(\lambda v) = J_{st} \circ \mathrm{d}(f \circ u)(0)\left(\frac{\partial}{\partial x}\right) = \mathrm{d}(f \circ u)(0)\left(J_{st}\frac{\partial}{\partial x}\right).$$

But since u is J-holomorphic one has

$$\mathrm{d}(f \circ u)(0)\left(J_{st}\frac{\partial}{\partial x}\right) = \mathrm{d}f \circ J \circ \mathrm{d}u(0)\left(\frac{\partial}{\partial x}\right) = \mathrm{d}f \circ J(\lambda v).$$

This implies the equality $\mathrm{d}f \circ J = \mathrm{d}f \circ J_{st}$.

However, such non-constant functions generally do not exist. Assume that f is a $\mathcal{C}^2$-function satisfying (1.6) and let a_{jk} be the entries of the matrix A. Then one may compute the following:

$$\frac{\partial^2 f}{\partial \bar{z}_l \partial \bar{z}_j} = \sum_{k=1}^{n}\left[\frac{\partial a_{jk}}{\partial \bar{z}_l} + \sum_{s=1}^{n} a_{sl}\frac{\partial a_{jk}}{\partial z_s}\right]\frac{\partial f}{\partial z_k} + \sum_{k=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 f}{\partial z_k \partial z_s}.$$

Thus the necessary conditions for the existence of such functions are

$$N_{jkl} = N_{jlk}, \quad N_{jkl} := \frac{\partial a_{jk}}{\partial \bar{z}_l} + \sum_{s=1}^{n} a_{sl}\frac{\partial a_{jk}}{\partial z_s}. \tag{1.7}$$

Furthermore, (1.7) assures the existence of n independent solutions forming a local diffeomorphism mapping into $\mathbb{C}^n$. Such a diffeomorphism is needed in order to provide integrability. However, the proof is much more difficult. One has to see that

$$L_j = \frac{\partial}{\partial \bar{z}_j} - \sum_{k=1}^{n} a_{jk}\frac{\partial}{\partial z_k}$$

form a basis of the space of $(0,1)$-vector fields and that every Lie bracket $[L_j, L_k]$ is again a $(0,1)$-vector field. The rest follows form the classical Newlander-Nirenberg theorem [**27**].

Remark 1.16. In the classical references this integrability condition is stated globally in terms of vanishing of a *Nijenhuis tensor*

$$N_J(X,Y) = [JX, JY] - J\,[JX, Y] - J\,[X, JY] - [X, Y]\,,$$

where $X, Y \colon M \to TM$ are real vector fields. See [**33**] for the relation.

1.3. Neighborhood of a point

In this section we discuss the normal forms for an almost complex structure J and its complex matrix A on a neighborhood of a point $p \in M$. One can always assume that there exists a chart φ around p such that $\varphi(p) = 0$ and $\varphi_*(J)(0) = J_{st}$ (Remark 1.6). However, due to non-integrability, it is in general impossible to find coordinates in which the structure would be standard on a neighborhood of the origin. Although, shrinking the neighborhood enough, one can see it as a small deformation of J_{st} on the unit ball $\mathbb{B} \subset \mathbb{R}^{2n}$ (small with derivatives). We use the usual $\mathcal{C}^1(\overline{\mathbb{B}})$-norm to express this (see § 2.1).

LEMMA 1.17. *Let (M, J) be an almost complex manifold. For every point $p \in M$ and $\lambda > 0$ there exist a neighborhood U of p and a diffeomorphism $\varphi\colon U \to \mathbb{B}$ such that $\varphi(p) = 0$, $\varphi_*(J)(0) = J_{st}$ and*

$$\|\varphi_*(J) - J_{st}\|_{\mathcal{C}^1(\overline{\mathbb{B}})} < \lambda.$$

PROOF. There exists a diffeomorphism φ defined on a neighborhood U' of p with the properties $\varphi(p) = 0$ and $\varphi_*(J)(0) = J_{st}$. For $K > 0$ consider the dilatations $\delta_K(z) = Kz$ on $\mathbb{R}^{2n}$. Set $\varphi_K = \delta_K \circ \varphi$. Then one has

$$\lim_{K\to\infty} \|(\varphi_K)_*(J) - J_{st}\|_{\mathcal{C}^1(\overline{\mathbb{B}})} = 0$$

We now set $U = \varphi_K^{-1}(\mathbb{B})$ for K large enough. □

Assume now that we work locally in the coordinates of the above lemma. Since J is close to J_{st} we have $\det(J + J_{st}) \neq 0$ on $\mathbb{B}$. Hence one can define its complex matrix A as in (1.4). The $\mathcal{C}^1$-norm of A is small and since $J(0) = J_{st}$ it vanishes at the origin. Let us normalize it even further. Since A is a matrix with complex entries, the usual componentwise partial derivatives make sense. The following result is due to Diedrich and Sukhov [**9**] (see also [**33**]). It shows that one can get rid of the complex derivatives at the origin by choosing an appropriate diffeomorphism.

THEOREM 1.18 (Diedrich-Sukhov normalization). *Let J be an almost complex structure defined on a neighborhood of the origin in $\mathbb{R}^{2n}$. Assume that $J(0) = J_{st}$ and let A be the complex matrix of J. There exists a quadratic change of coordinates φ fixing the origin, such that in the new coordinates $Z' = \varphi(Z)$ the direct image $A' = \varphi_*(A)$ has the properties $A'(0) = 0$ and $\frac{\partial A'}{\partial z'_j}(0) = 0$ for $j = 1, 2, \ldots, n$.*

PROOF. Let $\phi^l_{j,k} \in \mathbb{C}$. We set $\varphi(Z) = Z' = (z'_1, z'_2, \ldots, z'_n)$ to be

$$z'_l = z_l + \sum_{j,k=1}^{n} \phi^l_{j,k} z_j \bar{z}_k.$$

We denote by $\phi^{\bar{Z}}_j$ and ϕ^Z_k the matrices with entries $(\phi^l_{j,k})^n_{k,l=1}$ and $(\phi^l_{j,k})^n_{j,l=1}$ respectively. Using the notation from Lemma 1.15 one has

$$\varphi_Z(Z) = I + \sum_{k=1}^{n} \phi^Z_k \bar{z}_k \quad \text{and} \quad \varphi_{\bar{Z}}(Z) = \sum_{j=1}^{n} \phi^{\bar{Z}}_j z_j.$$

Since $A(0) = 0$ we have the expansion

$$A(Z) = \sum_{j=1}^{n} A_j^Z z_j + \sum_{k=1}^{n} A_k^{\bar{Z}} \bar{z}_k + O(|Z|^2),$$

where A_j^Z, $A_k^{\bar{Z}}$ are the corresponding $n \times n$ complex matrices.

We now apply the transformation rule from Lemma 1.15:

$$A'(\varphi(Z)) = \sum_{j=1}^{n} (A_j^Z - \phi_j^{\bar{Z}}) z_j + \sum_{k=1}^{n} A_k^{\bar{Z}} \bar{z}_k + O(|Z|^2).$$

Setting $\phi_j^{\bar{Z}} = A_j^Z$ we obtain $\frac{\partial A'}{\partial z_j}(0) = 0$. Note that this determines uniquely the quadratic part of φ. However, since $z'_j = z_j + O(|Z|^2)$ we also have vanishing of the derivatives with respect to z'_j. □

Looking at the proof and the transformation rule carefully, one can see that it was important for φ that it did not have a part linear in the variables $\bar{z}_l$. This would cause loss of the property $A(0) = 0$. On the other hand the mixed quadratic part $\sum_{j,k=1}^{n} \phi_{j,k}^l z_j \bar{z}_k$ does not affect conjugate derivatives of A. They can only be changed by applying a diffeomorphism with a part quadratic in the conjugate variable. However, we can not make them disappear in general.

THEOREM 1.19. *Let J be an almost complex structure defined on a neighborhood of the origin in $\mathbb{R}^{2n}$. Assume that $J(0) = J_{st}$ and let A be the complex matrix of J. The following statements are equivalent:*

(1) *There exists a quadratic coordinate diffeomorphism φ fixing the origin, such that in the new coordinates $Z' = \varphi(Z)$ the direct image $A' = \varphi_*(A)$ has the properties $A'(0) = 0$ and $\frac{\partial A'}{\partial \bar{z}'_j}(0) = 0$ for $j = 1, 2, \dots, n$.*
(2) *The entries a_{jk} of A satisfy the relation $\frac{\partial a_{jk}}{\partial \bar{z}_l}(0) = \frac{\partial a_{jl}}{\partial \bar{z}_k}(0)$, for every $j, l, k \in \{1, 2 \dots, n\}$.*

PROOF. Let $\phi_{j,k}^l \in \mathbb{C}$. We set $\varphi(Z) = Z' = (z'_1, z'_2, \dots, z'_n)$ to be

$$z'_l = z_l + \sum_{j,k=1}^{n} \phi_{j,k}^l \bar{z}_j \bar{z}_k.$$

We denote by $\phi_k^{\bar{Z}}$ the matrix with entries $(\phi_{j,k}^l + \phi_{k,j}^l)_{j,l=1}^n$. One has

$$\varphi_Z(Z) = I \quad \text{and} \quad \varphi_{\bar{Z}}(Z) = \sum_{j=k}^{n} \phi_k^{\bar{Z}} \bar{z}_k.$$

As before we use expansion of A and the transformation lemma 1.15:

$$A'(\varphi(Z)) = \sum_{j=1}^{n} A_j^Z z_j + \sum_{k=1}^{n}(A_k^{\bar{Z}} - \phi_k^{\bar{Z}})\bar{z}_k + O(|Z|^2).$$

To obtain vanishing of $\frac{\partial A'}{\partial \bar{z}_k}$ one needs $\phi_k^{\bar{Z}} = A_k^{\bar{Z}}$. But note that the matrices $\phi_k^{\bar{Z}}$ are symmetric. Thus (1) follows from (2).

Conversely, take an arbitrary quadratic diffeomorphism. It must have no part linear in $\bar{z}_l$ since otherwise we lose the property $A'(0) = 0$. Hence only by prescribing the $\sum_{j,k=1}^{n} \phi_{j,k}^l \bar{z}_j \bar{z}_k$ part one can affect the conjugate derivatives. We now repeat the above consideration. □

Remark 1.20. Note that the condition (2) in Theorem 1.19 is equivalent to the integrability conditions (1.7) when $A(0) = 0$. Indeed, being able to get rid of the conjugate linear part in A implies vanishing of the Nijenhuis tensor at a given point and vice versa [**33**].

1.4. Neighborhood of an embedded disc

Let u be a J-holomorphic embedding defined on $\mathbb{D}_\gamma := (1+\gamma)\mathbb{D}$ for some $\gamma > 0$. We will try to mimic the above results in a neighborhood of $u(\overline{\mathbb{D}})$. We include one original result (Theorem 1.25).

Since u is an embedding we can find a diffeomorphism mapping a neighborhood of $u(\mathbb{D}_\gamma)$ to $\mathbb{R}^{2n}$. Moreover, in the local coordinates one can always replace u with

$$u_0(\zeta) := (0, 0, \ldots, 0, \zeta), \quad \zeta \in \mathbb{D}_\gamma.$$

Recall also the initial local condition for J-holomorphicity (1.2) there. It can be interpreted in the following way: for each $\zeta \in \mathbb{D}_\gamma$ real vectors $\frac{\partial u}{\partial x}(\zeta)$ and $\frac{\partial u}{\partial y}(\zeta)$ form a $J(u(\zeta))$-invariant (real) two dimensional subspace of $T_{u(\zeta)}\mathbb{R}^{2n}$, that is, they are $J(u(\zeta))$-linearly dependent. As in [**19**] we will reparametrize u so that this dependence will become J_{st}.

Corollary 1.21. *Let (M, J) be a smooth almost complex manifold and u a J-holomorphic embedding defined on $\mathbb{D}_\gamma$. Assume that J and u are of class $\mathcal{C}^k$, $k \in \mathbb{N}$. There exists a $\mathcal{C}^k$-smooth diffeomorphism φ defined on the neighborhood of the image $u(\overline{\mathbb{D}})$ such that:*

$$\varphi \circ u(\zeta) = u_0(\zeta), \quad \varphi_*(J)(0, 0, \ldots, 0, \zeta) = J_{st}(0, , 0, \ldots, 0, \zeta), \quad \zeta \in \overline{\mathbb{D}}.$$

PROOF. We can find smooth vector fields $Y_1, \dots, Y_{n-1}$ on $\mathbb{R}^{2n}$ along $u(\mathbb{D}_\gamma)$ such that for every $\zeta \in \mathbb{D}_\gamma$, the set of vectors $\frac{\partial u}{\partial x}, Y_1, \dots Y_{n-1}$ are $J(u(\zeta))$-linearly independent. We define

$$\varphi(z_1, \dots, z_n) = u(z_n) + \sum_{j=1}^{n-1} z_j Y_j(u(z_n))$$

for $|z_n| < 1 + \gamma$ and $|z_j|$ small, $j < n$. □

Working locally with a structure continuously close to J_{st} allows us to define its complex matrix A as in (1.4). Since $J(u_0(\overline{\mathbb{D}})) = J_{st}$ it vanishes along the image $u_0(\overline{\mathbb{D}})$. But we prove in the sequel that, unlike in a neighborhood of a point, one can not assume its $\mathcal{C}^1$-norm to be small (a generalization of Lemma 1.17 is not possible). However, A can be approximated by a certain non-zero complex matrix.

We state this and all the following results in the space $\mathbb{C}^2$. It will greatly simplify the notation but still be sufficient to point out the main differences with respect to the previous section. Thus let $Z = (z, w)$ be the standard coordinates of $\mathbb{C}^2$ and $u_0(\zeta) = (0, \zeta) \in \mathbb{C}^2$ for $\zeta \in \overline{\mathbb{D}}$.

PROPOSITION 1.22. *Let J be an almost complex structure defined in a neighborhood of $u_0(\overline{\mathbb{D}})$. Assume that $J(u_0(\overline{\mathbb{D}})) = J_{st}$ and let A be the complex matrix of J with entries $a_{j,k}$, $j, k = 1, 2$. Set*

$$B_1(w) = \left(\frac{\partial a_{1,2}}{\partial z}\right)(0, w), \quad B_2(w) = \left(\frac{\partial a_{1,2}}{\partial \bar{z}}\right)(0, w),$$

and

$$A_1(z, w) = \begin{bmatrix} 0 & zB_1(w) + \bar{z}B_2(w) \\ 0 & 0 \end{bmatrix}. \tag{1.8}$$

Then given a neighborhood V of $u_0(\overline{\mathbb{D}})$ and $\lambda > 0$ there exists a diffeomorphism φ fixing the disc u_0 and mapping to V such that $A' = \varphi_*(A)$ has the properties $A'(u_0(\overline{\mathbb{D}})) = 0$ and $\|A' - A_1\|_{C^1(\overline{V})} \le \lambda$.

PROOF. Like in the proof of the Lemma 1.17, we consider dilatations that preserve the flat disc $u_0(\overline{\mathbb{D}})$ and shrink its neighborhood in the normal direction:

$$d_K \colon (z, w) \to (Kz, w)\,, \quad K > 0.$$

Let us now apply these dilatations using Lemma 1.15. The coefficients of A and its direct image A' are related as follows:

$$a'_{1,1}(z,w) = a_{1,1}\left(\frac{1}{K}z, w\right),$$

$$a'_{2,2}(z,w) = a_{2,2}\left(\frac{1}{K}z, w\right),$$

$$a'_{2,1}(z,w) = \frac{1}{K}a_{2,1}\left(\frac{1}{K}z, w\right),$$

$$a'_{1,2}(z,w) = Ka_{1,2}\left(\frac{1}{K}z, w\right).$$

Since $J(0,\zeta) = J_{st}$, the matrix $A(0,\zeta)$ vanishes, and thus the first three entries of the direct image A' also vanish in the limit as $K \to \infty$. In contrast, the last entry $a'_{1,2}$ of A' tends to the expression

$$z\frac{\partial a_{1,2}}{\partial z}(0,w) + \bar{z}\frac{\partial a_{1,2}}{\partial \bar{z}}(0,w).$$

This shows that for large $K > 0$ the structure A' is as close as desired to the structure A_1 given by (1.8). □

Our next goal is to obtain a normalization similar to the one from Diedrich and Sukhov in the case of a point. We would like the complex derivatives of A to vanish along the disc u_0. Note that if $A(0,\zeta) = 0$, vanishing of $\frac{\partial A}{\partial w}(0,\zeta)$ is immediate. It remains to eliminate the term $\frac{\partial A}{\partial z}(0,\zeta)$. We present here the Sukhov-Tumanov normalization [**37**].

THEOREM 1.23 (Sukhov-Tumanov normalization). *Let J be a $\mathcal{C}^3$-smooth almost complex structure defined on a neighborhood of $u_0(\overline{\mathbb{D}})$. Assume that $J(u_0(\overline{\mathbb{D}})) = J_{st}$ and let A be the complex matrix of J. There exists a $\mathcal{C}^2$-smooth diffeomorphism φ fixing the disc u_0 such that in the new coordinates $Z' = \varphi(Z)$ the direct image $A' = \varphi_*(A)$ has the properties $A'(u_0(\overline{\mathbb{D}})) = 0$ and $\frac{\partial A'}{\partial z'}(0,\zeta) = 0$ for every $\zeta \in \overline{\mathbb{D}}$.*

Remark 1.24. If J is $\mathcal{C}^k$-smooth, the structure corresponding to A' is of class $\mathcal{C}^{k-2}$. We do not claim that this is optimal for a given problem.

PROOF. Given functions f_j and g_j, $j = 1,2$, we set

$$\varphi(z,w) = (zf_1(w) + z\bar{z}f_2(w), w + zg_1(w) + z\bar{z}g_2(w)) = (z', w').$$

Note that φ is a diffeomorphism fixing u_0 if f_1 does not vanish on $\overline{\mathbb{D}}$. Further, by Lemma 1.15 it is preserving the property $A'(0,\zeta) = 0$ since one has $\varphi_{\overline{Z}}(0,\zeta) = 0$. We have

$$\frac{\partial A'}{\partial z} = \frac{\partial A'}{\partial z'}\frac{\partial z'}{\partial z} + \frac{\partial A'}{\partial \bar{z}'}\frac{\partial \bar{z}'}{\partial z} = (f_1 + \bar{z} f_2)\frac{\partial A'}{\partial z'} + \overline{\bar{z} f_2}\frac{\partial A'}{\partial \bar{z}'}.$$

Hence $\frac{\partial A'}{\partial z'}(0,\zeta)$ vanishes if and only if $\frac{\partial A'}{\partial z}(0,\zeta) = 0$. Deriving the transformation rule with respect to z we get

$$[A' \circ \varphi]_z = [\varphi_Z A - \varphi_{\bar{Z}}]_z \left(\overline{\varphi_Z} - \overline{\varphi_{\bar{Z}}} A\right)^{-1} + (\varphi_Z A - \varphi_{\bar{Z}}) \left[\left(\overline{\varphi_Z} - \overline{\varphi_{\bar{Z}}} A\right)^{-1}\right]_z.$$

Since $\varphi(0,\zeta) = (0,\zeta)$ and $\varphi_{\bar{Z}}(0,\zeta) = A(0,\zeta) = 0$ one has to prove that

$$\varphi_Z \frac{\partial A'}{\partial z} - \frac{\partial \varphi_{\bar{Z}}}{\partial z} = 0 \ \text{ on } \ u_0(\overline{\mathbb{D}}).$$

Note that

$$\varphi_Z(0,\zeta) = \begin{bmatrix} f_1(\zeta) & 0 \\ g_1(\zeta) & 1 \end{bmatrix}, \quad \frac{\partial \varphi_{\bar{Z}}}{\partial z}(0,\zeta) = \begin{bmatrix} f_2(\zeta) & (f_1)_{\bar{\zeta}} \\ g_2(\zeta) & (g_1)_{\bar{\zeta}} \end{bmatrix}.$$

Thus denoting by $b_{i,j}(\zeta)$ the coefficients of the matrix $\frac{\partial A'}{\partial z}(0,\zeta)$ we obtain the system

$$(f_1)_{\bar{\zeta}} - b_{1,2} f_1 = 0, \quad f_2 = b_{1,1} f_1,$$

$$(g_1)_{\bar{\zeta}} - b_{1,2} g_1 = b_{2,2}, \quad g_2 = b_{1,1} f_1 + b_{2,1}.$$

We find its $\mathcal{C}^2$-smooth solutions in § 2.2. □

This normalization allows us to simplify the matrix A_1 from (1.8). Indeed, if we allow B_2 to change, one can make B_1 vanish. However, note that by (1.7) the vanishing of B_2 is equivalent to the integrability of the structure corresponding to A_1. In general, one can not get rid of the conjugate term. Moreover, its modulus is a biholomorphic invariant. The precise result is stated in the following theorem which is the only original result in this chapter [**22**].

THEOREM 1.25. *Let J be a $\mathcal{C}^3$-smooth almost complex structure defined in a neighborhood of $u_0(\overline{\mathbb{D}})$. Assume that $J(u_0(\overline{\mathbb{D}})) = J_{st}$. Define B_1 and B_2 as in Proposition 1.22. Then the following hold:*

(1) *The structure A_1 from Proposition 1.22 is up to a diffeomorphism equivalent to the* model structure

$$A_\beta(z,w) = \begin{bmatrix} 0 & \bar{z}\beta(w) \\ 0 & 0 \end{bmatrix},$$

where a $\mathcal{C}^1$-smooth complex function β depends on B_1 and B_2.

(2) *Two model structures* A_β *and* $A_{\beta'}$ *are equivalent along* $u_0(\overline{\mathbb{D}})$ *if and only if there exists a non vanishing function* g *holomorphic on the neighborhood of* $\overline{\mathbb{D}}$ *and such that* $\bar{g}\beta' = g\beta$.

(3) *The modulus of* β *is invariant under any change of coordinates fixing* u_0 *and preserving the property* $J(0,\zeta) = J_{st}(0,\zeta)$.

Proof. Let $\varphi(z,w) = (z',w')$ be an arbitrary diffeomorphism in a neighborhood of $u_0(\overline{\mathbb{D}})$ that fixes u_0 and the property $J = J_{st}$ along $u_0(\overline{\mathbb{D}})$. This is expressed by the conditions

$$\varphi_Z = \begin{bmatrix} z'_z & 0 \\ w'_z & 1 \end{bmatrix}, \qquad \varphi_{\bar{Z}} = 0 \tag{1.9}$$

along $u_0(\overline{\mathbb{D}})$. Let us simplify the notation by setting $u' = \varphi(u)$, $A = A(u)$ and $A' = A'(u')$. Take the transformation rule from Lemma 1.15 and use the simplified notation:

$$A'\left(\overline{\varphi_Z} - \overline{\varphi_{\bar{Z}}}A\right) = \left(\varphi_Z A - \varphi_{\bar{Z}}\right).$$

Let τ be one of the variables $z, \bar{z}$. Differentiating with respect to τ we obtain the following matrix identity along u_0:

$$A'_\tau \overline{\varphi_Z} = \varphi_Z A_\tau - (\varphi_{\bar{Z}})_\tau. \tag{1.10}$$

We now restrict our attention to the $(1,2)$-entry of this matrix equation. Applying (1.9) we see that the following relation holds along u_0 :

$$(a'_{1,2})_\tau = z'_z (a_{1,2})_\tau - (z'_{\bar{w}})_\tau.$$

The matrix A' is expressed in the coordinates (z',w'), and the chain rule gives

$$\begin{aligned}
(a'_{1,2})_z &= z'_z (a'_{1,2})_{z'} + \bar{z}'_z (a'_{1,2})_{\bar{z}'} + w'_z (a'_{1,2})_{w'} + \bar{w}'_z (a'_{1,2})_{\bar{w}'}, \\
(a'_{1,2})_{\bar{z}} &= z'_{\bar{z}} (a'_{1,2})_{z'} + \bar{z}'_{\bar{z}} (a'_{1,2})_{\bar{z}'} + w'_{\bar{z}} (a'_{1,2})_{w'} + \bar{w}'_{\bar{z}} (a'_{1,2})_{\bar{w}'}.
\end{aligned}$$

Since $A'(0,\zeta) = 0$ (as the coordinate diffeomorphism preserves the property $J(0,\zeta) = J_{st}$), we have $A'_{w'}(0,\zeta) = A'_{\bar{w}'}(0,\zeta) = 0$ along u_0. By (1.9) we see that, along u_0, the system equals to

$$\begin{aligned}
(a'_{1,2})_z &= z'_z (a'_{1,2})_{z'} \\
(a'_{1,2})_{\bar{z}} &= \overline{z'_z} (a'_{1,2})_{\bar{z}'}.
\end{aligned}$$

If we now use (1.10) for $\tau = z$ and $\tau = \bar{z}$, we get

$$z'_z (a'_{1,2})_{z'} = z'_z (a_{1,2})_z - (z'_z)_{\bar{w}}, \tag{1.11}$$

$$\overline{z'_z} (a'_{1,2})_{\bar{z}'} = z'_z (a_{1,2})_{\bar{z}}. \tag{1.12}$$

To complete the proof of the first implication, we must make the term $(a'_{1,2})_{z'}$ vanish. By (1.11) this means we have to solve the equation $f_{\bar\zeta} - fB_1 = 0$, for $f(\zeta) = z'_z(0,\zeta)$. We again postpone this to the next chapter § 2.2. If we use the diffeomorphism $(z, w) \to (zf(w), w)$, we obtain the desired form of A_β for $\beta = B_2 f/\bar f$.

Consider now the equation (1.11). Suppose that $(a_{1,2})_z$ already vanishes along $u_0(\overline{\mathbb{D}})$. To preserve this property $(z'_z)_{\bar w}$ must also vanish along u_0. Setting $g(\zeta) = z'_z(0,\zeta)$, this implies the vanishing of $g_{\bar\zeta}$. So the equivalence relation is obtained by using coordinates proposed above (replacing f with g) for a non vanishing holomorphic function g. This concludes the proof of part (2).

Part (3) follows trivially from the fact that (1.12) is valid for every change of coordinates as in the theorem. □

Remark 1.26. Note that using Proposition 1.12 results can be stated in terms of an almost complex structure instead of its complex matrix.

As already remarked, results analogous to the above theorems and proposition hold in higher dimension. In case of Proposition 1.22 we use the same dilatations to obtain a matrix A_1 in which only the last column (without the diagonal entry) is nontrivial and the entries are linear in the normal variables, that is, B_1, B_2 are $(n-1) \times (n-1)$ matrix functions of z_n. Also a generalization of the Sukhov-Tumanov normalization is possible [**33**].

In addition to Sukhov-Tumanov normalization we could obtain vanishing of $\frac{\partial a_{1,1}}{\partial \bar z}$ and $\frac{\partial a_{2,1}}{\partial \bar z}$ along $u_0(\overline{\mathbb{D}})$. However, it is impossible to eliminate the term $\frac{\partial a_{2,2}}{\partial \bar z}$ in general; this term can only be made sufficiently small by using dilatations as in Proposition 1.22. Thus the complex matrix A_β is not a normal form of A. It can only be considered close to a given complex matrix.

CHAPTER 2

Solving the linearized problem

In this chapter recall some well known topics from the elliptic equations theory. We shall see in the sequel that solving the linear problem is indeed the crucial step in our work. We restrict ourselves to the case of *generalized analytic vectors*, that is, solutions of the equation

$$v_{\bar{\zeta}} + B_1 v + B_2 \bar{v} = 0.$$

A rather complete theory for these systems in the one-dimensional (scalar) case was given by Vekua [**39**] who called solutions of the above homogeneous systems the *generalized analytic functions*. In the same spirit, generalized analytic vectors were introduced by Pascali [**30**].

The following sections contain no original result, they only sum the work of several authors together. Firstly, for the sake of completeness and secondly, because these systems (and the corresponding boundary problems) have mostly been studied as a particular case of more general elliptic problems. We present only the parts relevant for the theory of J-holomorphic discs. We omit a large part of the proofs but include some rarely cited results of Habetha [**16**], Buchanan [**4**], and a recent work from Sukhov and Tumanov [**36**].

2.1. Preliminaries

This preliminary section is included for convenience of beginners. The topics are considered to be 'well known enough' to only label the statements and give no proofs or explicit references.

2.1.1. Banach spaces. A *Banach space* is a normed vector space which is complete in the metric induced by the norm. Let X and Y be Banach spaces. A linear operator $L\colon X \to Y$ is *bounded* if there exists $M > 0$ such that for all v in X

$$\|Lv\|_Y \leq M\|v\|_X.$$

We call the smallest number satisfying this condition *the norm* of L. We denote it by $\|L\|$ (it depends on the norms of X and Y).

The operator L is bounded if and only if it is continuous with respect to the norm of X and Y. If L is a bounded operator mapping X to itself with $\|L\| < 1$, then the operator $I_X - L$ is invertible, and its inverse is given by

$$(I_X - L)^{-1} = \sum_{k=0}^{\infty} L^k.$$

Here I_X stands for the identity map on X.

A bounded linear operator $K\colon X \to Y$ is *compact* if the image of any bounded subset of X is a relatively compact subset of Y. In particular, it suffices to prove that for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$ the sequence $(Lx_n)_{n\in\mathbb{N}} \subset Y$ contains a Cauchy subsequence.

A bounded linear operator $L\colon X \to Y$ is called *Fredholm* if its kernel and cokernel are finite-dimensional. Let us fix the notation for

$$\beta(L) = \dim\ker L \text{ and } \gamma(L) = \operatorname{codim}\operatorname{range}L.$$

The *Fredholm index* of L is defined by $\nu(L) = \beta(L) - \gamma(L)$. When L is Fredholm and K a compact operator, then $T + L$ is Fredholm and $\nu(L+K) = \nu(L)$. In particular, if K is a compact operator mapping the Banach space X to itself, then $I_X + K$ is Fredholm with $\nu(I_X + K) = 0$.

THEOREM 2.1 (Fredholm alternative). *Let K be a compact operator mapping the Banach space X to itself. If $I_X + K$ is injective then it is surjective as well.*

We often introduce a real *inner product*. It is defined as a symmetric bilinear form $\langle\, .\,,\, .\,\rangle : X \times X \to \mathbb{R}$ such that $\langle x, x\rangle \geq 0$ for every $x \in X$ and the equality holds if and only if $x = 0$. Given a bounded operator $L\colon X \to X$, one can, using the Riesz representation theorem, define a unique operator L^* such that

$$\langle Lx, y\rangle = \langle x, L^*y\rangle \quad \text{for every} \quad x, y \in X.$$

We call L^* the *adjoint operator* of L. It is a matter of a simple computation to show that $\ker L^* = (\operatorname{range}L)^{\perp}$, that is, for every $x \in X$: $L^*x = 0$ if and only if $\langle Ly, x\rangle = 0$ for every $y \in X$.

We extend this notion to a product of Banach spaces $X \times Y$. A continuous bilinear form $\langle\, .\,,\, .\,\rangle : X \times Y \to \mathbb{R}$ is called *regular* if the following statement holds: If $\langle x, y\rangle = 0$ for every $y \in Y$ then $x = 0$ and if $\langle x, y\rangle = 0$ for every $x \in Y$ then $y = 0$.

Assume we have two regular forms $\langle\, .\, ,\, .\,\rangle$ and $\langle\langle\, .\, ,\, .\,\rangle\rangle$ defined on $Y_1 \times X_2$ and $X_1 \times Y_2$. The operators $L\colon X_1 \to Y_1$ and $L^*\colon X_2 \to Y_2$ are *adjoint* with respect to these forms if

$$\langle Lx_1, x_2\rangle = \langle\langle x_1, L^* x_2\rangle\rangle\,,\ \text{for every } x_1 \in X_1,\ x_2 \in X_2.$$

We will need the following proposition.

PROPOSITION 2.2. *The equality $\nu(L) = -\nu(L^*)$ is equivalent to the following statement: $Lx_1 = y_1$ is solvable if and only if $\langle y_1, x_2\rangle = 0$ for every $x_2 \in \ker L^*$ and $L^*x_2 = y_2$ is solvable if and only if $\langle\langle x_1, y_2\rangle\rangle = 0$ for every $x_1 \in \ker L$.*

PROOF. The statement concerning solvability yields $\beta(L) = \gamma(L^*)$ and $\beta(L^*) = \gamma(L)$. Hence $\nu(L) = -\nu(L^*)$ follows trivially.

Conversely, let $x_2 \in X_2$ and $\pi_{x_2}(y_1) := \langle y_1, x_2\rangle$. Since the form is regular, the linear map $\Pi\colon x_2 \mapsto \pi_{x_2}$ is injective when mapping from X_2 to the dual space of functionals Y_1^*. Further, let $x_2 \in \ker L^*$. We have for every $x_1 \in X_1$

$$\langle\langle x_1, L^*x_2\rangle\rangle = \langle Lx_1, x_2\rangle = \pi_{x_2}(Lx_1) = 0$$

This embeds $\ker L^*$ into $\{\pi_{x_2} \in Y_1^*;\ \pi_{x_2}(Lx_1) = 0 \text{ for every } x_1 \in X_1\}$. By an algebraic argument, the dimension of this last space corresponds to $\gamma(L)$. Thus we have $\beta(L^*) \le \gamma(L)$ and we obtain $\beta(L) \le \gamma(L^*)$ similarly. Hence

$$\beta(L) \le \gamma(L^*) = \beta(L^*) - \nu(L^*) = \beta(L^*) + \nu(L) \le \gamma(L) + \nu(L) = \beta(L)$$

so that $\beta(L) = \gamma(L^*)$ and $\beta(L^*) = \gamma(L)$. Thus the above embedding is onto. This means that $Lx_1 = y_1$ is solvable if and only if $\pi_{x_2}(y_1) = \langle y_1, x_2\rangle = 0$ for all $x_2 \in \ker L^*$. We may obtain a similar result introducing π_{x_1} respectively. □

Let $U \subset X$ be an open subset. A map $F\colon U \to Y$ is called *Fréchet differentiable at $x \in U$* if there exists a bounded linear operator $A_x\colon X \to Y$ such that

$$\lim_{h\to 0} \frac{\|F(x+h) - F(x) - A_x(h)\|}{\|h\|} = 0.$$

We write $DF(x) = A_x$ and call it the *Fréchet derivative of F at x*. A function is said to be continuously differentiable (or $\mathcal{C}^1$) on U if it is differentiable at every $x \in U$ and the derivative $x \mapsto DF(x)$ is continuous.

THEOREM 2.3 (Inverse function theorem). *Let X and Y be Banach spaces and $F\colon X \to Y$ a $\mathcal{C}^1$-map. If $x_0 \in X$ and $DF(x_0)\colon X \to Y$ is a bounded linear isomorphism of X onto Y, then there exist open neighborhoods U of x_0 and V of $F(x_0)$ and a continuously differentiable map $G\colon V \to U$ such that $F \circ G = I_U$ and $G \circ F = I_V$.*

Suppose we have a map $f\colon X \times Y \to Z$ where X, Y and Z are Banach spaces. The partial derivative $D_Y f(x, y)$ in $(x, y) \in X \times Y$ is defined as a map $k \mapsto Df(x, y)(0, k)$, where $(0, k) \in \{0\} \times Y \subset X \times Y$.

THEOREM 2.4 (Implicit function theorem). *Let X, Y, Z be Banach spaces and $F\colon X \times Y \to Z$ a $\mathcal{C}^1$-map. If $(x_0, y_0) \in X \times Y$, $F(x_0, y_0) = 0$, and $D_Y F(x_0, y_0)$ is an isomorphism mapping Y onto Z, then there exist neighborhoods U of x_0 and V of y_0 and a continuously differentiable map $G\colon U \to V$ such that $F(x, G(x)) = 0$ and $F(x, y) = 0$ if and only if $y = G(x)$, for all $(x, y) \in U \times V$.*

We use this last theorem to derive a convenient corollary. A closed subspace Y of a Banach space X is called *complemented*, if there exists a closed subspace $Z \subset X$ such that $X = Y \oplus Z$.

COROLLARY 2.5. *Let X, Y be Banach spaces and $F\colon X \to Y$ a continuously Fréchet differentiable map. Let $x_0 \in X$ and $F(x_0) = 0$. Assume that the map $DF(x_0)\colon X \to Y$ is onto and that $\ker(DF)(x_0) \subset X$ is complemented. Then the following set is a $\mathcal{C}^1$-submanifold of X*

$$\mathcal{M} = \{x \in X : \ x \text{ near } x_0, \ f(x) = 0\}.$$

PROOF. Identify $X \cong \ker(DF)(x_0) \times W$ and write $x_0 = (x_0^{\ker}, x_0^W)$. Since on W one has $(D_W F)(x_0) = (DF)(x_0)$, it follows that $(D_W F)(x_0)$ is an isomorphism between W and Y. This implies existence of neighborhoods P_1 of $x_0^{\ker} \in \ker(DF)(x_0)$ and P_2 of $x_0^W \in W$ and a differentiable map $\Phi\colon P_1 \to P_2$ such that $F(x) = 0$ is equivalent to $x^W = \Phi(x^{\ker})$ for every $x = (x^{\ker}, x^W) \in P_1 \times P_2$. □

We conclude by stating an extension of the Brouwer fixed point theorem. A subset K of a vector space X is said to be *convex* if, for every $x, y \in K$ and $t \in [0, 1]$, we have $tx + (1 - t)y \in K$.

THEOREM 2.6 (Schauder fixed point theorem). *Let K be a convex subset of the Banach space X and $T\colon K \to K$ a continuous mapping such that $T(K)$ is contained in a compact subset of K. Then there exists a point $x \in K$ such that $Tx = x$.*

2.1.2. Spaces of functions. We introduce several vector spaces of complex functions defined on an open set $\Omega \subset \mathbb{C}$. We equip some of them with a norm so that they become Banach. We denote by $\zeta = x + iy \in \mathbb{C}$ the complex variable. For $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}_0^2$ and $|\gamma| = \gamma_1 + \gamma_2$ we define

$$D^\gamma f = \frac{\partial^{|\gamma|}}{\partial x^{\gamma_1} \partial y^{\gamma_2}} f.$$

Let $k \in \mathbb{N}_0$. The space $\mathcal{C}^k(\Omega)$ consists of complex functions whose partial derivatives up to the k-th order are continuous on Ω. We write $\mathcal{C}(\Omega) = \mathcal{C}^0(\Omega)$. The class $\mathcal{C}^\infty(\Omega)$ is the intersection of all spaces $\mathcal{C}^k(\Omega)$, $k \in \mathbb{N}_0$. By $\mathcal{H}(\Omega) \subset \mathcal{C}^\infty(\Omega)$ we denote the closed space of all holomorphic functions. To obtain a Banach space we assume that Ω is relatively compact with smooth boundary and consider the subspace $\mathcal{C}^k(\overline{\Omega})$ consisting of functions whose derivatives are continuous up to the boundary. The norm of $f \in \mathcal{C}^k(\overline{\Omega})$ is defined by

$$\|f\|_{\mathcal{C}^k(\overline{\Omega})} = \sum_{|\gamma| \leq k} \max_{\zeta \in \overline{\Omega}} |D^\gamma f(z)|.$$

Let $0 < \alpha < 1$. Given a complex function f on Ω let us assume that there exist a constant $M \in \mathbb{R}$ such that for every $\zeta_1, \zeta_2 \in \Omega$

$$|f(\zeta_1) - f(\zeta_2)| \leq M |\zeta_1 - \zeta_2|^\alpha.$$

Then f is said to be *Hölder continuous with exponent α on Ω*. We denote by $\mathcal{C}^{k,\alpha}(\Omega)$ the subset of $\mathcal{C}^k(\Omega)$ consisting of functions whose partial derivatives up to order k are Hölder continuous with exponent α on Ω. We write $\mathcal{C}^\alpha(\Omega)$ for $\mathcal{C}^{0,\alpha}(\Omega)$. We call them *Hölder spaces.* Taking $\zeta_1, \zeta_2 \in \overline{\Omega}$ one can define the Hölder spaces on the closure as well. In particular, if Ω is relatively compact they are Banach with respect to the norm

$$\|f\|_{\mathcal{C}^{k,\alpha}(\overline{\Omega})} = \|f\|_{\mathcal{C}^k(\overline{\Omega})} + \sum_{|\gamma| \leq k} \max_{\zeta_1, \zeta_2 \in \overline{\Omega}} \frac{|D^\gamma f(\zeta_1) - D^\gamma f(\zeta_2)|}{|\zeta_1 - \zeta_2|^\alpha}.$$

The inclusions $\mathcal{C}^{k+1,\alpha}(\overline{\Omega}) \subset \mathcal{C}^{k+1}(\overline{\Omega}) \subset \mathcal{C}^{k,\alpha}(\overline{\Omega})$ are obviously bounded. Furthermore, for $0 \leq \alpha < \beta \leq 1$ the inclusion $\mathcal{C}^\beta(\overline{\Omega}) \subset \mathcal{C}^\alpha(\overline{\Omega})$ is bounded and the bound depends on the diameter of Ω :

$$\|f\|_{\mathcal{C}^\alpha(\overline{\Omega})} \leq \operatorname{diam}(\Omega)^{\beta - \alpha} \|f\|_{\mathcal{C}^\beta(\overline{\Omega})}. \tag{2.1}$$

Note that a sequence $(f_n)_{n \in \mathbb{N}}$ bounded in any Hölder norm is equicontinous. Hence by Arzelà-Ascoli theorem, this last inclusion is compact.

Let $p \geq 1$ and $f \colon \Omega \to \mathbb{C}$ measurable. We define

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(\zeta)|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}.$$

The definition may be extended for $p = \infty$ by

$$\|f\|_{L^\infty(\Omega)} := \inf \{ M > 0; \ |f(\zeta)| \leq M \ \text{ almost everywhere on } \Omega \}.$$

The *Lebesgue space* consist of all measurable functions whose $L^p(\Omega)$-norm is bounded. They are all Banach, but they are not closed for the product of functions. Indeed, the Hölder inequality states that, if $f_j \in L^{p_j}(\Omega)$ for $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1$, then $f_1 f_2 \in L^p(\Omega)$:

$$\|f_1 f_2\|_{L^p(\Omega)} \leq \|f_1\|_{L^{p_1}(\Omega)} \|f_2\|_{L^{p_2}(\Omega)}. \tag{2.2}$$

If Ω is relatively compact and $1 \leq p < q \leq \infty$ then the inclusions $L^q(\Omega) \subset L^p(\Omega)$ are bounded. The bound depends on the area:

$$\|f\|_{L^q(\Omega)} \leq \operatorname{area}(\Omega)^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^p(\Omega)}. \tag{2.3}$$

Obviously, the inclusion $\mathcal{C}(\overline{\Omega}) \subset L^p(\Omega)$ is also bounded in this case.

The integration by parts formula yields that for $f \in \mathcal{C}^k(\Omega)$ and all smooth compactly supported functions $\varphi \in \mathcal{C}_c^\infty(\Omega)$ one has

$$\int_\Omega f \, D^\gamma \varphi \, \mathrm{d}\mu = (-1)^{|\gamma|} \int_\Omega \varphi \, D^\gamma f \, \mathrm{d}\mu \text{ for } |\gamma| \leq k.$$

The left-hand side of this equation still makes sense if we only assume f to be locally integrable. Assume that there exists a locally integrable function v on Ω, such that

$$\int_\Omega f \, D^\gamma \varphi \, \mathrm{d}\mu = (-1)^{|\gamma|} \int_\Omega \varphi \, v \, \mathrm{d}\mu,$$

for every $\varphi \in C_c^\infty(\Omega)$. We call v the *weak γ-th partial derivative of f*, or the derivative in the Sobolev sense. If it exists, it is uniquely determined almost everywhere on Ω. On the other hand, when $f \in \mathcal{C}^k(\Omega)$, the classical and the weak derivative coincide. Thus, we use $D^\gamma f$ to denote also the weak derivatives. The *Sobolev spaces* $W^{k,p}(\Omega)$ are defined by

$$W^{k,p}(\Omega) = \{ f \in L^p(\Omega) \colon D^\gamma f \in L^p(\Omega), \ \forall |\gamma| \leq k \}.$$

The norm is defined as the sum of all L^p-norms:

$$\|f\|_{\mathcal{W}^{k,p}(\Omega)} = \sum_{|\gamma| \leq k} \|D^\gamma f\|_{L^p(\Omega)}.$$

Further, we define the weak partial derivatives with respect to ζ and $\bar{\zeta}$ with the usual relations. Due to regularity of solutions of elliptic partial differential equation we have the following Weyl-type theorem.

THEOREM 2.7. *Let $\Omega \subset \mathbb{C}$ be a domain. If $u \in W^{1,1}(\Omega)$ and $u_{\bar{\zeta}} = 0$ in the Sobolev sense then $u \in \mathcal{H}(\Omega)$.*

We conclude with the classical Sobolev embedding theorem stating that the existence of sufficiently many weak derivatives implies the existence and continuity of the classical derivatives.

THEOREM 2.8 (Sobolev embedding theorem). *Let $\Omega \subset \mathbb{C}$ be a relatively compact open subset and $(k-r-\alpha)/2 = 1/p$. Then the embedding $W^{k,p}(\Omega) \subset \mathcal{C}^{r,\alpha}(\overline{\Omega})$ is bounded.*

We will mostly use these definitions in the case when $\Omega = \mathbb{D}$. When dealing with vector functions $u \colon \overline{\mathbb{D}} \to \mathbb{C}^n$, the required conditions are supposed to hold componentwise. When $n > 1$, an additional index will be used $L^p(\mathbb{D})^n$, $W^{k,p}(\mathbb{D})^n$, $\mathcal{C}^k(\overline{\mathbb{D}})^n$ or $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})^n$. We have already used the componentwise $\mathcal{C}^1$-norm for matrix functions in the first chapter.

2.1.3. Birkhoff factorization. At the end of this chapter, we study the Riemann-Hilbert problem concerning functions defined on $\overline{\mathbb{D}}$ and satisfying a certain boundary condition. We present here the standard matrix decomposition that will be used.

Let $f \colon \partial\mathbb{D} \to \mathbb{C}$ be a continuous nowhere-vanishing function. The *winding number* of f is the integer representing the total number of times that the image of $\partial\mathbb{D}$ winds counterclockwise around the origin:

$$W(f) = \frac{1}{2\pi} \oint_{\partial\mathbb{D}} \mathrm{d}(\arg f).$$

The following two relations are obvious: $W(fg) = W(f) + W(g)$ and $W(\bar{f}) = W(1/f) = -W(f)$. Furthermore, if f is continuous and nowhere-vanishing on $\overline{\mathbb{D}}$, then $W(f) = 0$.

Let us denote by $GL(n,\mathbb{C})$ the set of all invertible $n \times n$ complex matrices. The *Birkhoff matrix decomposition theorem* states the following: given a matrix function $B \colon \partial\mathbb{D} \to GL(n,\mathbb{C})$ one can write

$$B(\zeta) = F_1(\zeta)\Lambda(\zeta)F_2(\zeta), \quad \zeta \in \partial\mathbb{D},$$

where the function $F_1 \colon \overline{\mathbb{D}} \to GL(n,\mathbb{C})$ is smooth and holomorphic on $\mathbb{D}$, the function $F_2 \colon \mathbb{C} \cup \{\infty\} \setminus \mathbb{D} \to GL(n,\mathbb{C})$ is smooth and holomorphic on $\mathbb{C} \cup \{\infty\} \setminus \overline{\mathbb{D}}$, and

$$\Lambda(\zeta) = \operatorname{diag}\left(\zeta^{\kappa_1}, \zeta^{\kappa_2}, \dots, \zeta^{\kappa_n}\right). \tag{2.4}$$

The integers $\kappa_j \in \mathbb{Z}$ are called *partial indices* and are invariant up to a permutation. Hence we usually arrange them to be decreasing in size from the largest κ_1 to the smallest κ_n. We denote by $\kappa = \sum_{j=1}^{n} \kappa_j$ the *total index*. We will frequently use the following corollary.

COROLLARY 2.9. *Let $B\colon \partial\mathbb{D} \to GL(n, \mathbb{C})$ be a matrix function such that $B = \overline{B}^{-1}$. Then there exists a map $\Theta\colon \overline{\mathbb{D}} \to GL(n, \mathbb{C})$, smooth and holomorphic on $\mathbb{D}$, such that $\Theta\Lambda\bar{\Theta}^{-1} = B$ on $\partial\mathbb{D}$, where Λ is defined as in* (2.4)*. The total index κ equals $W(\det B)$.*

PROOF. Let $B = F_1 \Lambda F_2$ be the Birkhoff decomposition. We define $F_j^*(z) := F_j(1/\bar{z})$, $j = 1, 2$. Obviously one has $F_j(\zeta) = F_j^*(\zeta)$ for the points $\zeta \in \partial\mathbb{D}$. Hence on the boundary

$$B = F_1 \Lambda F_2 = \overline{B}^{-1} = \overline{F_2}^{-1} \Lambda \overline{F_1}^{-1} = \overline{F_2^*}^{-1} \Lambda \overline{F_1^*}^{-1}.$$

But note that F_2^* and F_1^* are smooth and holomorphic on the open sets $\mathbb{D}$ and $\mathbb{C} \cup \{\infty\} \setminus \overline{\mathbb{D}}$ respectively. Thus, due to the uniqueness of the decomposition, $F_2 = \overline{F_1^*}^{-1}$. Hence on the boundary $F_2 = \overline{F_1}^{-1}$. Finally, note that since $\Theta = F_1$ is invertible, one has $W(\det \Theta) = 0$ and $W(\det B) = W(\det \Lambda) = \kappa$. $\square$

2.2. Singular integral operators

In this section we introduce two classical integral operators. Due to the nature of this survey we define them on the unit disc, although a more general theory is known. The reader may find it in the monography of Vekua [**39**] along with the proofs of the properties that we list below (Proposition 2.10 and Proposition 2.11). We apply them to prove the existence of solutions for the two differential equations arising in the first chapter: the Beltrami equation and a scalar elliptic equation from the proofs of Theorem 1.23 and Theorem 1.25.

2.2.1. Cauchy-Green transform. We denote by $\zeta = x + iy$ the complex variable on $\overline{\mathbb{D}}$ and by z an element of $\mathbb{C}$. For $u \in \mathcal{C}^1(\overline{\mathbb{D}})$ the generalized Cauchy's integral formula yields

$$u(z) = -\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{u(\zeta)\mathrm{d}\zeta}{z - \zeta} + \frac{1}{\pi} \iint_{\mathbb{D}} \frac{\partial u}{\partial \bar{z}}(\zeta) \frac{\mathrm{d}x\mathrm{d}y}{z - \zeta}. \tag{2.5}$$

Since the fist summand is holomorphic, the second one contains all the information on the $\bar{\partial}$-derivative. We apply it to solve an inhomogeneous Cauchy-Riemann equation $\frac{\partial u}{\partial \bar{\zeta}} = f$ on $\mathbb{D}$. If $f \in \mathcal{C}(\overline{\mathbb{D}})$, the general solution is given by

$$u = \phi + \frac{1}{\pi} \iint_{\mathbb{D}} \frac{f(\zeta)}{z - \zeta} \mathrm{d}x \mathrm{d}y.$$

where ϕ is holomorphic on $\mathbb{D}$ [**17**].

This motivates us to define the *Cauchy-Green transform* of a complex valued function $u \colon \mathbb{D} \to \mathbb{C}$ by

$$T(u)(z) = \frac{1}{\pi} \iint_{\mathbb{D}} \frac{u(\zeta)}{z - \zeta} \, \mathrm{d}x \, \mathrm{d}y. \tag{2.6}$$

It is well defined for $u \in L^p(\mathbb{D})$, $p > 1$. If $1 < p < 2$, then $T(u) \in L^k(\mathbb{D})$ for $k = \frac{2p}{p-2}$. Moreover, $v(z) = Tu(z)$ makes sense also for $z \in \mathbb{C}$ and vanishes at infinity. If $p > 2$, then $T(u) \in \mathcal{C}^{1-2/p}(\mathbb{C})$. We collect some other properties in the following proposition.

Proposition 2.10. *Let $p > 2$, $0 < \alpha < 1$ and $k \in \mathbb{N}_0$:*

(1) *The operator $u \mapsto T(u)$ is well defined and bounded mapping from $L^p(\mathbb{D})$ to $\mathcal{C}^{1-2/p}(\overline{\mathbb{D}})$ or from $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$ to $\mathcal{C}^{k+1,\alpha}(\overline{\mathbb{D}})$.*
(2) *If $u \in L^p(\mathbb{D})$, then $T(u) \in W^{1,p}(\mathbb{C})$ and the weak derivative $[T(u)]_{\bar{z}}$ equals u on $\mathbb{D}$ and vanishes on $\mathbb{C} \backslash \overline{\mathbb{D}}$.*
(3) *Let $B \in L^p(\mathbb{D})$ and $u \in L^r(\mathbb{D})$, $r > \frac{2p}{p-2}$. Then $Bu \in L^s(\mathbb{D})$ for $s = (1/p + 1/r)^{-1} > 2$. The operator $u \mapsto T(Bu)$ is well defined and compact on $L^r(\mathbb{D})$. If $B \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$, this same operator is compact when mapping the space $\mathcal{C}^{k+1,\alpha}(\overline{\mathbb{D}})$ to itself.*

Set $p > 2$, $0 < \alpha < 1$ and $k \in \mathbb{N}_0$. In the proof of Theorem 1.23 and Theorem 1.25 the following equation arises:

$$u_{\bar{\zeta}} + B_1 u = 0. \tag{2.7}$$

Let us assume that $B_1 \in L^p(\mathbb{D})$ and assume that the equation holds in the Sobolev sense. We seek a solution in the form $u = \phi e^v$ where ϕ denotes a function holomorphic in $\mathbb{D}$. We obtain

$$\phi e^v (v_{\bar{\zeta}} + B_1) = 0.$$

By parts (1) and (2) of the above proposition, we can solve this equation explicitly as $v = -T(B_1) \in \mathcal{C}^{1-2/p}(\overline{\mathbb{D}})$. Assuming that $B_1 \in \mathcal{C}^{k,\alpha}(\mathbb{D})$ the solution $u = \phi e^{-TB_1}$ of (2.7) belongs to $\mathcal{C}^{k+1,\alpha}(\mathbb{D})$. To obtain continuity up to the boundary, one needs $B_1 \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$ and $\phi \in \mathcal{C}^{k+1,\alpha}(\overline{\mathbb{D}})$.

In order to complete the proof of Theorem 1.23 one also needs to prove the existence of a solution for the non-homogeneous problem

$$u_{\bar{\zeta}} + B_1 u = f.$$

Let us assume that $f, B \in L^p(\mathbb{D})$. By part (2) it is enough to consider

$$u + T(B_1 u) = Tf.$$

Since (3) yields that $u \mapsto T(B_1 u)$ is compact, one can use Fredholm alternative (Theorem 2.1) on the space $L^r(\mathbb{D})$. Assume that $v \in L^r(\mathbb{D})$ and $v + T(B_1 v) = 0$. By part (3) v is also bounded and Hölder continuous with index $1-2/r$ on $\mathbb{C}$. Thus by part (2) the function $\phi = ve^{TB_1}$ is entire, bounded and vanishing at the infinity. Hence by Liouville theorem both ϕ and v are trivial. Thus there exists a solution $u \in L^r(\mathbb{D})$. Since $u = Tf - T(B_1 u)$ the solution is indeed Hölder continuous with index $1-2/r$ on $\overline{\mathbb{D}}$. Assuming that $f, B_1 \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$, one can repeat this argument until reaching $u \in \mathcal{C}^{k+1,\alpha}(\overline{\mathbb{D}})$.

We stress that in the first chapter B_1 and f came from the derivatives of a certain complex matrix function. It belonged to an almost complex structure that was assumed to be of class $\mathcal{C}^3$ at least. Hence they are both of class $\mathcal{C}^2$ and solutions are of class $\mathcal{C}^{2,\alpha}$ for every $0 < \alpha < 1$. Note also that the zero sets of u and ϕ coincide in the homogeneous case. Hence a discrete zero set can be prescribed for u. We seek in the proof a non-vanishing function.

However, reading the proofs carefully, one can note that we indeed need functions solving the above equations on some small neighborhood of the closed unit disc. To obtain them one should apply the Cauchy-Green transform defined on $\mathbb{D}_\gamma := (1+\gamma)\mathbb{D}$ for some $\gamma > 0$.

2.2.2. Beltrami equation. Next, we define the *Ahlfors-Beurling transform*. It arises as a derivation of T with respect to the complex variable. The operator is singular and has to be considered as the Cauchy principal value.

$$\Pi(h)(z) = \lim_{\epsilon\to 0} \frac{1}{\pi} \iint_{\mathbb{D}\setminus\{|\zeta - z| < \epsilon\}} \frac{u(\zeta)}{(z-\zeta)^2}\, \mathrm{d}x\, \mathrm{d}y. \tag{2.8}$$

This operator will enable us to solve the Beltrami equation from the first chapter. Again, $v(z) = \Pi(u)(z)$ is a continuous function defined for $z \in \mathbb{C}$. It is holomorphic on $\mathbb{C}\setminus\overline{\mathbb{D}}$ and it vanishes at infinity. Some other properties are listed in the proposition below.

PROPOSITION 2.11. *Let $p > 2$, $0 < \alpha < 1$ and $k \in \mathbb{N}_0$.*

(1) *There exists a constant $C_p > 0$ such that $\|\Pi(h)\|_p < C_p \|h\|_p$. Furthermore, we can choose C_p tending to 1 as p tends to 2.*
(2) *For $u \in L^p(\mathbb{D})$ we have $[T(u)]_z = \Pi(u)$ in the Sobolev sense.*
(3) *The operator $\Pi\colon \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}}) \to \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$ is linear and bounded.*

Part (1) is often called the *Calderon-Zygmund inequality.*

For $|\mu| < q < 1$ we consider the Beltrami equation

$$\varphi_{\bar\zeta} = \mu\varphi_\zeta. \tag{2.9}$$

Assume that $\mu \in L^p(\mathbb{D})$. We claim that for an appropriate $p > 2$ one can solve the following equation

$$h - \Pi(\mu h) = \Pi\mu.$$

Indeed, by Calderon-Zygmund inequality we have $\|\Pi(\mu h)\|_p < qC_p$. Since $q < 1$ and C_p tends to 1, we may choose $p > 2$ such that $qC_p < 1$. The left hand side can then be viewed as an invertible linear operator mapping h to $\Pi\mu$. One may formally write the solution in the form

$$h = \Pi(\mu) + \Pi(\mu\Pi(\mu)) + \Pi(\mu\Pi(\mu\Pi(\mu))) + \dots.$$

We further define

$$\varphi(z) = z + T(\mu h)(z) + T(\mu)(z). \tag{2.10}$$

Parts (2) of Proposition 2.11 and Proposition 2.10 imply that

$$\varphi_{\bar\zeta} = \mu(h+1) = \mu\left(1 + \Pi(\mu h) + \Pi(\mu)\right) = \mu\varphi_\zeta.$$

By construction $h \in L^p(\mathbb{D})$ and hence $\varphi \in \mathcal{C}^{1-2/p}(\overline{\mathbb{D}}) \cap W^{1,p}(\mathbb{D})$. If we assume that μ belongs to $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$, the same is true for h (it is a convergent sum) and φ belongs to $\mathcal{C}^{k+1,\alpha}(\overline{\mathbb{D}})$. However, the function μ arising from the equation in the first chapter is $\mathcal{C}^\infty$-smooth. Thus the equation (2.9) has a smooth solution.

Further, we would like to prove its invertibility. We give here a classical argument of Ahlfors [**1**]. Note that the solution φ defined as in (2.10) has the following properties: it solves (2.9) on $\mathbb{D}$; it is continuous on $\mathbb{C}$; it is holomorphic outside $\overline{\mathbb{D}}$. We claim that there exists no non-trivial function $f\colon \mathbb{C} \to \mathbb{C}$ that would, in addition to the above properties, satisfy the following two assumptions: f vanishes at infinity and $f_\zeta \in L^p(\mathbb{C})$. Indeed, the first assumption implies vanishing of the holomorphic function $\phi = f - T(f_\zeta)$ at infinity. Thus by a classical argument $f = T(f_\zeta)$. Further the Calderon-Zygmund inequality yields

$$\|f_\zeta\|_{L^p(\mathbb{D})} = \|\Pi(\mu f_{\bar\zeta})\|_{L^p(\mathbb{D})} = \|\Pi(\mu f_\zeta)\|_{L^p(\mathbb{D})} \leq C_p q \|f_\zeta\|_{L^p(\mathbb{D})}.$$

Let us set $p > 2$ so that $C_p q < 1$. In this space the norm $\|f_\zeta\|_{L^p(\mathbb{D})}$ has to be equal to zero. Hence f_ζ and $f_{\bar\zeta}$ vanish on $\mathbb{D}$. This along with holomorphicity on $\mathbb{C}\backslash\overline{\mathbb{D}}$ and vanishing of f at infinity implies $f \equiv 0$.

The above fact gives uniqueness when solving

$$f_{\bar\zeta} = \mu f_\zeta + \sigma, \quad \sigma \in L^p(\overline{\mathbb{D}}). \tag{2.11}$$

There exists precisely one function $f : \mathbb{C} \to \mathbb{C}$ with the following properties: f solves (2.11) on $\mathbb{D}$, f is holomorphic outside $\overline{\mathbb{D}}$, f vanishes at infinity, and $f_\zeta \in L^p(\mathbb{D})$. Let $k \in L^p(\mathbb{D})$ be a solution of the equation $k - \Pi(\mu k) = \Pi(\sigma)$. The unique solution of (2.11) equals:

$$f_\sigma = T(\mu k) + T(\mu\sigma). \tag{2.12}$$

We now return to the smooth solution φ defined in (2.10). Regarded as a real map to $\mathbb{R}^2$ its Jacobian is given by

$$\mathrm{Jac}(\varphi) = |\varphi_\zeta|^2 - |\varphi_{\bar\zeta}|^2 = |\varphi_\zeta|^2(1 - |\mu|^2).$$

We prove that it is nowhere vanishing. Let us differentiate the equation (2.9) with respect to ζ :

$$(\varphi_\zeta)_{\bar\zeta} = \mu(\varphi_\zeta)_\zeta + \mu_\zeta\varphi_\zeta.$$

We write $\varphi_\zeta = e^f$. It follows that

$$f_{\bar\zeta} = \mu f_\zeta + \mu_\zeta.$$

We have proved that this non-homogeneous Beltrami equation admits a unique solution f_{μ_ζ} satisfying the above listed assumptions. Check that $\log(\varphi_\zeta)$ satisfies them as well. Thus by uniqueness $\varphi_\zeta = e^{f_{\mu_\zeta}}$. Hence

$$\mathrm{Jac}(\varphi) = |e^{f_{\mu_\zeta}}|^2(1 - |\mu|^2) > 0.$$

Since φ induces a smooth map of the Riemann sphere into itself which is locally a diffemorphism, φ must be a diffeomorphism. Indeed, it must be onto by connectedness of the sphere, since its image is an open and closed subset; but then, as a covering map, φ must cover each point of the sphere the same number of times. Since only ∞ is sent to ∞, it follows that φ is one-to-one.

2.3. Generalized analytic functions and vectors

In this section we study the Carleman-Bers-Vekua system:

$$u_{\bar{\zeta}} + B_1 u + B_2 \bar{u} = 0. \tag{2.13}$$

We assume that $B_1, B_2 \in L^p(\mathbb{D})$ and allow the equation to hold in the Sobolev sense. The theory is based on the methods similar to the ones we have used in solving (2.7) (the case when $B_2 = 0$). The difference is that we are unable to give an explicit solution in general, even in the homogeneous case. Still we can discuss their properties by representing them in an appropriate form related to the usual holomorphic solutions.

2.3.1. Generalized analytic functions. A rather complete theory for these systems in the one-dimensional (scalar) case was given by Vekua [**39**]. We follow his work. He named the solutions of the above homogeneous equation *generalized analytic functions*.

THEOREM 2.12 (Similarity principle). *Let $B_1, B_2 \in L^p(\mathbb{D})$, $p > 2$, and T the Cauchy-Green transform defined in (2.6). Let $u \in W^{1,p}(\mathbb{D})$ be a solution of (2.13). There exists a holomorphic function $\phi \in \mathcal{H}(\mathbb{D})$ such that $u = \phi e^{-T(g)}$ for*

$$g(z) = \begin{cases} B_1(z) + B_2(z)\frac{\overline{u(z)}}{u(z)}, & u(z) \neq 0; \\ B_1(z) + B_2(z), & u(z) = 0. \end{cases}$$

PROOF. Let us define $\phi = ue^{T(g)}$. Since u solves (2.13) by properties of the Cauchy-Green transform (Proposition 2.10) the weak derivative $\phi_{\bar{\zeta}}$ vanishes where $u(z) \neq 0$. Further,

$$\frac{1}{\pi}\iint_{\mathbb{D}} \phi\, \varphi_{\bar{\zeta}}\, \mathrm{d}x\mathrm{d}y = \frac{1}{\pi}\iint_{\mathbb{D}\setminus\{u=0\}} \phi\, \varphi_{\bar{\zeta}}\, \mathrm{d}x\mathrm{d}y = 0$$

for every smooth and compactly supported function $\varphi \in \mathcal{C}_c^\infty(\mathbb{D})$. Thus $\phi_{\bar{\zeta}} = 0$ in Sobolev sense on $\mathbb{D}$. By Theorem 2.7 the function ϕ is holomorphic in the usual sense. □

Using the theorem one can prove that the zero set of u is discrete. Moreover, an analog of the Liouville theorem can be obtained for solutions of (2.13) on $\mathbb{C}$ [**39**].

Set $p > 2$, $k \in \mathbb{N}_0$ and $0 < \alpha < 1$. We establish a further correspondence with holomorphic functions by introducing the operator

$$\Phi(u) := u + T(B_1 u + B_2 \bar{u}) = \phi. \tag{2.14}$$

If u solves (2.13), then ϕ is holomorphic in the usual sense. By part (3) of Proposition 2.10 we have: if $B_1, B_2 \in L^p(\mathbb{D})$ then the operator Φ is well defined and Fredholm on $L^r(\mathbb{D})$ for $r > \frac{2p}{p-2}$. If $B_1, B_2 \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$ the same is true on the space $\mathcal{C}^{k+1,\alpha}(\overline{\mathbb{D}})$. Hence if the coefficients B_1, B_2 are of class $L^p(\mathbb{D})$ (or $\mathcal{C}^{k,\alpha}(\mathbb{D})$) the solutions of (2.13) are in $\mathcal{C}^{1-2/r}(\mathbb{D})$ (or $\mathcal{C}^{k+1,\alpha}(\mathbb{D})$ respectively). Their continuity up to the boundary depends on the function ϕ.

Finally, we apply the Fredholm alternative (Theorem 2.1) to prove that Φ is surjective. If $v \in L^r(\mathbb{D})$ and $v + T(B_1 v + B_2 \bar{v}) = 0$, then v is holomorphic on $\mathbb{C}\backslash\overline{\mathbb{D}}$ and belongs to $\mathcal{C}^{1-2/r}(\mathbb{C})$ (Proposition 2.10). Hence if we take g from the Similarity principle the function $\phi = ve^{T(g)}$ is holomorphic on $\mathbb{D}$ and $\mathbb{C}\backslash\overline{\mathbb{D}}$, and Hölder continuous (hence bounded) on $\mathbb{C}$. Since it also vanishes at the infinity, this implies $v = \phi \equiv 0$ by a classical Liouville theorem.

PROPOSITION 2.13. *Operator Φ defined as in (2.14), mapping the space $L^r(\mathbb{D})$ (or $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$) to itself, is bounded and one-to-one.*

2.3.2. Generalized analytic vectors. In the same spirit *generalized analytic vectors* were introduced by Pascali [**30**]. Thus, when dealing with vector function $u = (u_1, \ldots, u_n)$, $n > 1$, and $n \times n$ matrix functions B_1,B_2, the system (2.13) is often called *Pascali system*. We consider from now on the Cauchy-Green transform (2.6) to be defined for vector functions mapping componentwise. We do not change the notation since the dimension should be clear from the context.

It is a natural question if the results analogous to the scalar case hold for $n \geq 2$. When considering the Similarity principle, the answer is affirmative although Pascali originally proved only a local version.

THEOREM 2.14 (Local Similarity principle). *Let B_1, B_2 be $n \times n$ complex matrix functions with entries of class $L^p(\mathbb{D})$, $p > 2$. Suppose that $u \in W^{1,p}(\mathbb{D})^n$ is a solution of (2.13). Let $\zeta_0 \in \overline{\mathbb{D}}$. There exist a neighborhood $U \subset \mathbb{C}$ of ζ_0, a nonsingular matrix function S continuous on $U \cap \overline{\mathbb{D}}$, and a vector function ϕ holomorphic on $U \cap \mathbb{D}$ and continuous on $U \cap \overline{\mathbb{D}}$ such that $u = S\phi$.*

Remark 2.15. The original result was only for $\zeta_0 \in \mathbb{D}$ but we shall need this version from [**38**] in what follows. We stress also that Pascali's result was improved by Buchanan [**3, 5**] giving a global version.

PROOF. Let us denote by u_j the entries of u and by b_{ij}^k the entries of B_k, $k = 1, 2$. Let H be the matrix with entries $h_{ij} = b_{ij}^1 + b_{ij}^2 \bar{u}_j / u_j$. We rewrite the equation (2.13) into

$$u_{\bar{\zeta}} + Hu = 0.$$

We fix $\zeta_0 \in \overline{\mathbb{D}}$ and for $\epsilon > 0$ put $\mathbb{D}_\epsilon = \left\{\zeta \in \overline{\mathbb{D}};\ |\zeta - \zeta_0| \leq \epsilon\right\}$. Define $H_\epsilon = H$ for $\zeta \in \mathbb{D}_\epsilon$ and zero elsewhere. We claim that there exists a unique solution $w \in L^\infty(\mathbb{D})^n$ of the equation

$$w + T(H_\epsilon w) = e_j, \tag{2.15}$$

for $e_1, e_2, \dots e_n$ the standard basis vectors in $\mathbb{C}^n$. Indeed, consider a term $h_{ij} w_j$, where w_j denotes an entry of w. It belongs to the space $L^p(\mathbb{D})$. Hence $T(h_{ij} w_j) \in \mathcal{C}^{1-2/p}(\overline{\mathbb{D}})$ by Proposition 2.10. Further, by Hölder spaces inclusions (2.1) we have for $\alpha = 1 - 2/p$:

$$\|T(h_{ij} w_j)\|_{\mathcal{C}(\mathbb{D}_\epsilon)} \leq \epsilon^\alpha \|T(h_{ij} w_j)\|_{\mathcal{C}^\alpha(\mathbb{D}_\epsilon)} \leq \epsilon^\alpha \|T(h_{ij} w_j)\|_{\mathcal{C}^\alpha(\overline{\mathbb{D}})}.$$

But since T is bounded, there exists $C > 0$ such that

$$\|T(h_{ij} w_j)\|_{\mathcal{C}^\alpha(\overline{\mathbb{D}})} \leq C \|h_{ij} w_j\|_{L^p(\mathbb{D})} \leq C \|h_{ij}\|_{L^p(\mathbb{D})} \|w_j\|_{L^\infty(\mathbb{D})}.$$

Hence the operator $w \mapsto w + T(H_\epsilon w)$ is uniformly close to the identity in $L^\infty(\mathbb{D})^n$ and thus invertible.

Let $s_j \in L^\infty(\mathbb{D})^n$ be a solution of the above equation (2.15) for e_j. Because of T they indeed belong to spaces $W^{1,p}(\mathbb{D})^n$ and $\mathcal{C}^{1-2/p}(\overline{\mathbb{D}})^n$. Let S be the matrix with columns $s_1, s_2, \dots, s_n$. When ϵ tends to zero, S tends to the identity. For ϵ small enough we may define $\phi = S^{-1} u$. By differentiating (2.15) we obtain $S_{\bar{\zeta}} + HS = 0$ in the interior of $\mathbb{D}_\epsilon$. We now have

$$u_{\bar{\zeta}} + Hu = S_{\bar{\zeta}} \phi + S \phi_{\bar{\zeta}} + HS\phi = S\phi_{\bar{\zeta}} = 0.$$

Hence $\phi_{\bar{\zeta}} = 0$ on the interior of $\mathbb{D}_\epsilon$ in Sobolev sense. By Theorem 2.7 this implies holomorphicity. Note that $\phi \in W^{1,p}(\mathbb{D})$ implies its Hölder continuity with index $1 - 2/p$ up to the boundary (Theorem 2.8). Thus the theorem holds for $\zeta_0 \in \partial\mathbb{D}$ as well. □

Similarly as before one can define an integral operator Φ analogous to (2.14). It provides the same regularity correspondence between the matrices B_1 and B_2 and a given generalized vector in the spaces $L^p(\mathbb{D})^n$ and $\mathcal{C}^{k,\alpha}(\mathbb{D})^n$. However, although it remains Fredholm, the Fredholm alternative can not be applied in general since Φ may have a nontrivial kernel. We illustrate this using an example of Habetha [**16**] (see [**4**]).

Example 2.16. For $u = (u_1, u_2)$ consider on $\mathbb{D}$ the system

$$u_{\bar\zeta} + \begin{pmatrix} 0 & 6\zeta^2/(3-\zeta^2\bar\zeta^2) \\ -1 & 0 \end{pmatrix} u = 0.$$

Let us differentiate the second row by $\bar\zeta$ and plug it into the first one:

$$(u_2)_{\bar\zeta\bar\zeta} = -(u_1)_{\bar\zeta} = \frac{6\zeta^2}{\zeta^2\bar\zeta^2 - 3} u_2.$$

Observe that $v(\zeta) = \bar\zeta - \frac{1}{3}\zeta^2\bar\zeta^3$ solves the above equation. The other particular solution is given by $\psi(\zeta) = \sum b_k \zeta^{2k}\bar\zeta^{2k}$, where the coefficients b_k solve the recursion relation

$$b_0 = 1, \quad b_k = \prod_{j=1}^{k} \frac{(j-1)(2j-3)-3}{3(2j-1)j}.$$

Since $0 < b_k < \frac{1}{3^k}$ for $k \geq 2$, this series converges for $|\zeta| \leq 3$, as do its derivatives. This gives a general solution of the original equation:

$$\begin{bmatrix} u_1(\zeta) \\ u_2(\zeta) \end{bmatrix} = \begin{bmatrix} \psi_{\bar\zeta}(\zeta)\phi_1(\zeta) + (1-\zeta^2\bar\zeta^2)\phi_2(\zeta) \\ \psi(\zeta)\phi_1(\zeta) + (\bar\zeta - \frac{1}{3}\zeta^2\bar\zeta^3)\phi_2(\zeta) \end{bmatrix}, \tag{2.16}$$

where ϕ_1, ϕ_2 are holomorphic on $\mathbb{D}$. In particular, for $\zeta \in \partial\mathbb{D}$

$$\begin{bmatrix} u_1(\zeta) \\ u_2(\zeta) \end{bmatrix} = \begin{bmatrix} \lambda_1\zeta\phi_1(\zeta) \\ \lambda_2\phi_1(\zeta) + \frac{2}{3}\bar\zeta\phi_2(\zeta) \end{bmatrix}, \tag{2.17}$$

where $\lambda_1 = 2\sum_{k=0}^{\infty} k b_k$ and $\lambda_2 = \sum_{k=0}^{\infty} b_k$.

Recall now the Cauchy integral formula (2.5). A vector function $w = (w_1, w_2)$ belongs to $\ker\Phi$ if and only if for every $z \in \mathbb{D}$

$$C(w_j)(z) := \oint_{\partial\mathbb{D}} \frac{w_j(\zeta)\mathrm{d}\zeta}{\zeta - z} = 0, \quad j \in \{1, 2\}.$$

However, we know that $C(\phi)(z) = \phi(z)$ when ϕ is holomorphic. Hence in our particular case (2.17) such a condition is fulfilled when $\phi_1 \equiv 0$ and ϕ_2 is a complex constant. Hence $\dim_{\mathbb{R}} \ker\Phi = 2$.

Remark 2.17. Note that the solution $w(\zeta) = (1-\zeta^2\bar\zeta^2, \bar\zeta - \frac{1}{3}\zeta^2\bar\zeta^3)$ can be extended holomorphicaly outside the unit disc by $v(\zeta) = (0, \frac{2}{3\zeta})$. In particular, it vanishes at infinity. The author gave this example to point out that the Liouville theorem can not be generalized for a vector function solving the system (2.13) on $\mathbb{C}$ when $n \geq 2$. Such a statement is crucial when proving triviality of the kernel.

However, the above example does not contradict the existence of a one-to-one correspondence between the generalized analytic and the usual holomorphic vector functions. It was proved in the recent paper of Sukhov and Tumanov [**36**] that one can modify the correspondence (2.14) for $n \geq 2$ and obtain bijectivity. To do so, a small holomorphic term depending on the coefficients B_1 and B_2 has to be added.

Before stating the exact result, let us introduce the real inner product of vector functions $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$

$$\langle u, v \rangle = \frac{1}{2} \sum_{j=1}^{n} \iint_{\mathbb{D}} \mathrm{Re}\,(u_j \bar{v}_j)\, \mathrm{d}x \mathrm{d}y. \tag{2.18}$$

It is well defined on $L^1(\mathbb{D})^n$. By Hölder inequality (2.2) we may regard it as a bilinear form acting on $L^r(\mathbb{D})^n \times L^q(\mathbb{D})^n$, $q = r/(r-1)$. Let us find an operator adjoint to Φ from (2.14) with respect to this product.

First note that $\langle \bar{u}, v \rangle = \langle u, \bar{v} \rangle$ and that $\langle B_j u, v \rangle = \left\langle u, \overline{B_j}^T v \right\rangle$. Further, by reordering the integrals one gets $T^*(u) = -\overline{T(\bar{v})}$. Thus

$$\Phi^*(v) = v - \overline{B_1^T T(\bar{v})} - B_2^T T(\bar{v}). \tag{2.19}$$

We now use the fact that $(\mathrm{range}(\Phi))^\perp = \ker(\Phi^*)$ in the proof below.

Theorem 2.18. *Let B_1, B_2 be $n \times n$ matrix functions with entries of class $L^p(\mathbb{D})$, $p > 2$. Let Φ be defined as in (2.14) on $L^r(\mathbb{D})^n$, $r > \frac{2p}{p-2}$. We denote by N the real dimension of $\ker(\Phi)$ and by $w_1, w_2, \ldots, w_N$ its basis. There exist arbitrarily small vectors $p_1, p_2, \ldots, p_N \in \mathcal{H}(\mathbb{D})^n$ such that the operator*

$$\tilde{\Phi}(h) := \Phi(h) + \sum_{j=1}^{N} \langle h, w_j \rangle\, p_j$$

is bounded and has trivial kernel. In particular, the inverse $\tilde{\Phi}^{-1}$ is bounded. The function $\phi = \tilde{\Phi}(h)$ is holomorphic if and only if h satisfies $h_{\bar{\zeta}} + B_1 h + B_2 h = 0$.

Proof. Suppose $v \in \ker(\Phi^*)$. By definition we have $v \in L^q(\mathbb{D})^n$ for $1 < q < 2$. But since $v = \overline{B_1^T T(\bar{v})} + B_2^T T(\bar{v})$, we can improve this. The properties of T yield that $T(\bar{v}) \in L^k(\mathbb{D})^n$ for $k = 2q/(2-q) > 2$. Since $B_j \in L^p(\mathbb{D})$, by Hölder inequality (2.2) v belongs to $L^{k'}(\mathbb{D})^n$ where

$$k' = \frac{1}{\frac{1}{p} + \frac{1}{k'}} = \frac{k}{1 - k(\frac{1}{2} - \frac{1}{p})} < \frac{k}{1 - (\frac{1}{2} - \frac{1}{p})} = \beta k \text{ for } 1 < \beta < 2.$$

We apply this argument finitely many times until reaching $v \in L^p(\mathbb{D})^n$.

The function $u = \overline{T(\bar{v})} \in W^{1,p}(\mathbb{D})^n$ is a generalized analytic vector satisfying the equation

$$u_{\bar{\zeta}} - \overline{B_1^T} u - B_2^T \overline{u} = 0.$$

Fix $\zeta_0 \in \partial\mathbb{D}$. We have $\zeta \to \frac{1}{\zeta_0 - \zeta} \in \mathcal{H}(\mathbb{D})$. Hence if $v \in (\mathcal{H}(\mathbb{D})^n)^{\perp}$ then $u = \overline{T(\overline{v})} = 0$ on $\partial\mathbb{D}$. Let U, S and ϕ be defined for u and ζ_0 as in the Theorem 2.14. By the usual reflection principle ϕ can be extended across the boundary. Since it vanishes on $\partial\mathbb{D} \cap U$, it has to vanish on U as well. Thus u is trivial on U. We now cover $\overline{\mathbb{D}}$ with a finite number of neighborhoods on which the local Similarity principle can be applied. Note that the holomorphic vector has to vanish on every neighborhood intersecting with U. We repeat such an argument until reaching $u \equiv 0$ on $\overline{\mathbb{D}}$. Hence the intersection $(\mathcal{H}(\mathbb{D})^n)^{\perp} \cap \mathrm{Ker}(\Phi^*)$ is trivial.

Since $(\mathrm{range}(\Phi))^{\perp} = \mathrm{Ker}\Phi^*$ we have

$$\mathcal{H}(\mathbb{D})^n + \mathrm{range}(\Phi) = L^r(\mathbb{D})^n.$$

There exist $p_1, p_2, \ldots, p_N \in \mathcal{H}(\mathbb{D})^n$ such that

$$\mathrm{Span}_{\mathbb{R}}(p_1, p_2, \ldots, p_N) \oplus \mathrm{range}(\Phi) = L^r(\mathbb{D})^n. \tag{2.20}$$

We define the modified operator $\tilde{\Phi}\colon L^r(\mathbb{D})^n \to L^r(\mathbb{D})^n$ as

$$\tilde{\Phi}(u) = \Phi(u) + \sum_{j=1}^{N} \langle u, w_j \rangle \, p_j.$$

Suppose that $\tilde{\Phi}(u) = 0$. It follows from (2.20) that $\Phi(u) = 0$ and $\langle u, w_j \rangle = 0$ for all $j = 1, 2, \ldots, N$. Since the w_j's form a basis of $\ker(\Phi)$ over $\mathbb{R}$ we get $u = 0$. Thus $\tilde{\Phi}$ is one to one and the function $\phi = \tilde{\Phi}(u)$ is holomorphic if and only if u is a generalized analytic vector. $\square$

Remark 2.19. If the entries of B_1, B_2 belong to $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$, $k \geq 0$, $0 < \alpha < 1$, then $\tilde{\Phi}$ is an invertible bounded operator of the space $\mathcal{C}^{k+1,\alpha}(\overline{\mathbb{D}})$. Furthermore, since the linear independence is stable under small perturbations, p_j's can be chosen to be polynomials. Indeed, holomorphic functions can be approximated by polynomials, hence the fact that the range of a Fredholm operator is closed implies the rest.

Remark 2.20. Note that the one-to-one correspondence between generalized analytic and the usual holomorphic vectors assures unique solvability of the non-homogeneous problem

$$u_{\bar{\zeta}} + B_1 u + B_2 \bar{u} = f. \tag{2.21}$$

Indeed, the solution is given by $u = \tilde{\Phi}^{-1}(Tf)$. Suppose that the entries of B_1, B_2 and f belong to $L^p(\mathbb{D})$. Then u is automatically in $\mathcal{C}^{1-2/r}(\overline{\mathbb{D}})^n$ (Proposition 2.10). If the coefficients are even $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$-regular, one can, by repeating such a consideration, reach $u \in \mathcal{C}^{k+1,\alpha}(\overline{\mathbb{D}})$.

Remark 2.21. We often want the holomorphic vector ϕ and a given generalized analytic solution u to have the same center $u(0) = \phi(0)$. In order to achieve this, we introduce the normalized transform

$$T_0(u)(z) = \frac{1}{\pi} \iint_{\mathbb{D}} \left(\frac{u(\zeta)}{z-\zeta} - \frac{u(\zeta)}{\zeta} \right) \mathrm{d}x\, \mathrm{d}y. \tag{2.22}$$

In addition to the properties of the usual Cauchy-Green transform we have $T_0(u)(0) = 0$. Thus $\Phi_0(u)(0) = u(0)$, where Φ_0 is defined by

$$\Phi_0(u) = u + T_0(B_1 u + B_2 \bar{u}).$$

Again, the kernel of Φ_0 may be trivial when the dimension of the problem is $n \geq 2$. But one can, as in the theorem above, find the holomorphic vectors $p_1^0, p_2^0, \ldots, p_N^0$ satisfying $p_j^0(0) = 0$ and modify the correspondence Φ_0 to become invertible

$$\tilde{\Phi}_0(h) := \Phi_0(h) + \sum_{j=1}^{N} \langle h, w_j \rangle\, p_j^0,$$

Note that the proof has to be slightly changed. Firstly, the adjoint operator has to be reproduced (it differs from the one above but its generalized analytic nature remains). Secondly, one has to replace the set $\mathcal{H}(\mathbb{D})^n$ with a set of holomorphic vector functions vanishing at the origin. This assures that $\tilde{\Phi}_0(u)(0) = u(0)$. See [**36**, Section 3].

2.4. The Riemann-Hilbert problem for Pascali type systems

In this section we study solutions of the non-homogeneous Pascali system (2.21) that in addition satisfy a classical Riemann-Hilbert boundary condition. That is, given $P\colon \partial\mathbb{D} \to GL(n, \mathbb{C})$ and an n-dimensional real vector function ψ, we require that $\mathrm{Re}(Pu) = \psi$ is valid on $\partial\mathbb{D}$. Boundary problems of this type have been studied by several authors, mostly as particular cases of certain elliptic boundary problems; see Vekua [**39**] for the scalar case and Bojarski [**2**], Gilbert and Buchanan [**5**] and Wendland [**40**] for higher dimension. We present some parts of the theory that are relevant for our survey.

We denote by $\mathcal{C}^{k,\alpha}_{\mathbb{R}}(\partial\mathbb{D})^n$ the space of real vector functions that are Hölder continuous with index α on $\partial\mathbb{D}$. We investigate $R = (R_1, R_2)$ mapping from $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})^n$ to $\mathcal{C}^{k-1,\alpha}(\overline{\mathbb{D}})^n \times \mathcal{C}^{k,\alpha}_{\mathbb{R}}(\partial\mathbb{D})^n$ given by

$$R_1(v) = v_{\bar\zeta} + B_1 v + B_2 \bar v \quad \text{and} \quad R_2(v) = \operatorname{Re}(Pv)\,. \tag{2.23}$$

Assuming that $\det P \neq 0$, we are interested in the following dimensions:

$$\beta(R) = \dim\ker(R), \quad \gamma(R) = \operatorname{codim}\operatorname{range}(R). \tag{2.24}$$

The scalar holomorphic case is classical. The dimensions are known to depend on the winding number $W(P)$ (see for instance [**26**]).

THEOREM 2.22. *Suppose that $n = 1$ and $B_1, B_2 \equiv 0$. We have*

$$\beta(R) = \begin{cases} -2W(P) + 1 & \text{if } W(P) \leq 0; \\ 0 & \text{if } W(P) > 0; \end{cases}$$

$$\gamma(R) = \begin{cases} 0 & \text{if } W(P) \leq 0; \\ 2W(P) - 1 & \text{if } W(P) > 0. \end{cases}$$

It is obvious that R is Fredholm in this simplest case with the Fredholm index equal to $\nu(R) = -2W(P) + 1$. Our observations will be based on the fact that the property is preserved for $B_1, B_2 \neq 0$ and $n \geq 2$ (see [**2, 40**]). An explicit index formula similar to the one above was provided by Bojarski. We give here the proof from [**40**].

PROPOSITION 2.23.

$$\nu(R) = -2W(\det P) + n.$$

PROOF. The proof is based on the fact that the Fredholm index depends continuously on the operator. We construct a family of operators depending continuously on the parameter $t \in [0, 1]$ that takes the initial problem into a holomorphic one with diagonal boundary conditions.

Let us define R^t as in (2.23) but for $(B^t_j) := (1 - t)B_j,\ j = 1, 2$. It remains to construct a family of matrices P^t such that $\det P^t \neq 0$ for every $t \in [0, 1]$ and that for $t = 1$ the determinant of P^1 agrees with the initial one.

Since $\det P \neq 0$ the entry $P_{1j}(\zeta)$ is non zero for at least one $j \in \{1, 2, \ldots n\}$ depending on $\zeta \in \partial\mathbb{D}$. Since P_{1j} is continuous, we may find a partition of unity, namely smooth real-valued functions

$\chi'_{1,1}, \chi'_{1,2}, \ldots, \chi'_{1,n}$ such that with a suitable $t_j \in \{0,1\}$, $\chi_{1,j} = (-1)^{t_j}\chi'_{1,j}$

$$\tilde{P}_{11} = \sum_{j=1}^{n} \chi_{1j} P_{1j} \neq 0 \text{ on } \partial\mathbb{D}.$$

The functions $\chi_{1,j}$ can be completed into a system of smooth functions χ_{kj} such that $\det(\chi_{kj})_{k,j=1}^{n} = 1$. When solving $R(v) = (\psi, \phi)$ we may replace the entries ϕ_k by $\sum_{j=1}^{n} \chi_{kj}\phi_j$ and work with the boundary matrix $\tilde{P}$ whose entries are $\tilde{P}_{jk} = \sum_{l=1}^{n} \chi_{jl} P_{lk}$. One may choose t'_js in such a way that the determinant of $\tilde{P}$ agrees with $\det P$.

Applying the above procedure beforehand, we may assume that the initial problem has non zero diagonal entries P_{jj}. We will now eliminate the first row continuously by using P_{11} as a pivot. Let $\tau \in [0,1]$ and $t = \frac{\tau}{n+1}$. We set the entries of P^t to be

$$P^t_{j1} = P_{j1} \text{ and } P^t_{jk} = P_{jk} - \tau_1 \frac{P_{j1}}{P_{11}} P_{1k},\ k \geq 2.$$

For $t_1 = \frac{1}{n+1}$ one has $P^{t_1}_{k1} = 0$ and $\det P^{t_1} = \det P$. Repeating this construction n times, we obtain a lower triangular matrix for $t_n = \frac{n}{n+1}$.

The last step is to continuously perturb the boundary condition into a diagonal form. Let $\tau \in [0,1]$. For $t = \frac{n+\tau}{n+1}$ we set the off-diagonal entries P_{jk}, $k > j$, to be

$$P^t_{jk} = (1-\tau) P^{t_n}_{jk}.$$

Since the matrix P^1 is diagonal, the problem can be broken into n scalar holomorphic boundary problems. Furthermore, $\det P = \prod_{j=1}^{n} P^1_{jj}$. Hence by the above theorem

$$\nu(R) = \sum_{j=1}^{n} (-2W(P^1_{jj}) + 1) = -2W(\det P) + n.$$

□

However, the Fredholm index itself provides only a partial information on the dimensions $\beta(R)$ and $\gamma(R)$. We establish another way of computing them by introducing an operator adjoint to R. As in [**40**] we introduce the following two bilinear forms

$$\langle v, (\psi, \phi) \rangle = -\frac{1}{2} \sum_{j=1}^{n} \iint_{\mathbb{D}} \mathrm{Re}(v_j \psi_j) \mathrm{d}x\mathrm{d}y + \oint_{\partial\mathbb{D}} \phi^T \mathrm{Im}(Pv) \mathrm{d}s$$

$$\langle\langle (\psi, \phi), v \rangle\rangle = \frac{1}{2} \sum_{j=1}^{n} \iint_{\mathbb{D}} \mathrm{Re}(v_j \psi_j) \mathrm{d}x\mathrm{d}y - \oint_{\partial\mathbb{D}} \phi^T \mathrm{Im}\left(\frac{\partial \zeta}{\partial s} P^{-T} v\right) \mathrm{d}s$$

for every $v \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})^n$ and $(\psi, \phi) \in \mathcal{C}^{k-1,\alpha}(\overline{\mathbb{D}})^n \times \mathcal{C}^{k,\alpha}_{\mathbb{R}}(\partial\mathbb{D})^n$. Here $\zeta = x + iy \in \overline{\mathbb{D}}$ and s is the natural parameter on $\partial\mathbb{D}$. We define the operator $R^* = (R_1^*, R_2^*)$ by

$$R_1^*(v) = v_{\bar\zeta} - B_1^T v - \bar{B_2}^T \bar{v} \ \text{ on } \overline{\mathbb{D}}, \quad R_2^*(v) = \mathrm{Re}\left(\frac{\partial\zeta}{\partial s} P^{-T} v\right) \ \text{ on } \partial\mathbb{D}.$$

Its properties are listed in the following proposition.

PROPOSITION 2.24. *Let R be defined as in* (2.23). *For the two linear forms and its adjoint R^* defined above the following hold:*

(1) *For every $v, w \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$ we have $\langle\langle R(w), v\rangle\rangle = \langle w, R^*(v)\rangle$.*
(2) *We have $\nu(R^*) = -\nu(R)$, $\beta(R) = \gamma(R^*)$ and $\gamma(R) = \beta(R^*)$.*
(3) *The Riemann-Hilbert problem $R(v) = (\psi, \phi)$ is solvable if and only if $\langle\langle(\psi, \phi), w\rangle\rangle = 0$ for every $w \in \ker(R^*)$.*

PROOF. We begin with the part (1). The Green's formula yields

$$\iint_{\mathbb{D}} \mathrm{Re}(v_j (w_j)_{\bar\zeta}) \mathrm{d}x\mathrm{d}y = -\iint_{\mathbb{D}} \mathrm{Re}((v_j)_{\bar\zeta} w_j) \mathrm{d}x\mathrm{d}y + \frac{1}{i}\mathrm{Re}\left(\oint_{\partial\mathbb{D}} v_j w_j \mathrm{d}\zeta\right).$$

Further

$$\oint_{\partial\mathbb{D}} \mathrm{Re}\left(P^{-T}\frac{\partial\zeta}{\partial s} v\right)^T \mathrm{Im}(Pw)\mathrm{d}s + \oint_{\partial\mathbb{D}} \mathrm{Re}(Pw)^T \mathrm{Im}\left(P^{-T}\frac{\partial\zeta}{\partial s} v\right)\mathrm{d}s =$$

$$= \oint_{\partial\mathbb{D}} \mathrm{Im}\left(\frac{\partial\zeta}{\partial s} w^T v\right)\mathrm{d}s = -i\oint_{\partial\mathbb{D}} \mathrm{Re}\left(\frac{\partial\zeta}{\partial s} w^T v\right)\mathrm{d}s = \frac{1}{i}\mathrm{Re}\left(\oint_{\partial\mathbb{D}} w^T v \,\mathrm{d}\zeta\right)$$

To conclude one has to verify that $\langle\langle(\bar{v}, \phi), w\rangle\rangle = \langle\bar{w}, (\bar{v}, \phi)\rangle$ and that $\langle\langle(Bv, \phi), w\rangle\rangle = \left\langle w, (-B^T v, \phi)\right\rangle$.

Note that by Proposition 2.23 one has the identity

$$\nu(R^*) = -2W(\det P^T) - 2W\left(\frac{\partial\zeta}{\partial s}\right) + n = 2W(\det P) - n = -\nu(R)$$

If the forms $\langle\, .\, ,\, .\, \rangle$ and $\langle\langle\, .\, ,\, .\, \rangle\rangle$ are regular, the rest of (2) and part (3) follow from the Proposition 2.2.

Assume that $\langle v, (\psi, \phi)\rangle = 0$ for every argument on the right hand side. If one takes $\phi = 0$ and $\psi = \bar{v}$, this implies $v = 0$. Conversely, let the above be true for any $v \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$. Assume that $\psi_1(\zeta) \neq 0$ for $\zeta \in \mathbb{D}$. We fix v_1 to be a smooth function with the constant value $\overline{\psi_1(\zeta)}$ on a small neighborhood of ζ and vanishing on the complement of a slightly larger closed neighborhood. In particular, it vanishes on $\partial\mathbb{D}$. We assume all other v_j's to be trivial. We have $\langle v, (\psi, \phi)\rangle \neq 0$. Hence by contradiction $\psi \equiv 0$. Choosing v to be a smooth extension of the function $iP^{-1}\bar\phi$ to $\mathbb{D}$, we obtain vanishing of ϕ as well. $\square$

Recall now the Birkhoff factorization introduced in § 2.1.3. By Corollary 2.9 there exists a smooth map $\Theta\colon \overline{\mathbb{D}} \to GL(n,\mathbb{C})$, holomorphic on $\mathbb{D}$ and such that

$$\Theta\Lambda\bar{\Theta}^{-1} = -P^{-1}\bar{P} \quad \text{on } \partial\mathbb{D}.$$

Here $\Lambda(\zeta) = \operatorname{diag}(\zeta^{\kappa_1}, \zeta^{\kappa_2}, \dots, \zeta^{\kappa_n})$, $\kappa_j \in \mathbb{Z}$. Note that the total index $\kappa = -2\,W(\det P)$ is even.

Applying the factorization, our consideration can be restricted to the simplified version of the operator $R = (R_1, R_2)$ from (2.23):

$$R_1(v) = v_{\bar{\zeta}} + \tilde{B}_1 v + \tilde{B}_2 \bar{v} \quad \text{and} \quad R_2(v) = v - \Lambda\bar{v}. \tag{2.25}$$

The coefficients $\tilde{B}_1 = \Theta^{-1}B_1\Theta$ and $\tilde{B}_2 = \Theta^{-1}B_1\overline{\Theta}$ have changed under this holomorphic change of variables, but remain zero when one has $B_1 = B_2 = 0$. We proceed by giving some computations from Vekua's scalar theory [**39**].

Proposition 2.25. *Let $\beta(R)$ and $\gamma(R)$ be defined as in (2.24). Suppose that $n = 1$. We have*

$$\beta(R) = \begin{cases} \kappa + 1 & \text{if } \kappa \geq 0; \\ 0 & \text{if } \kappa < 0, \end{cases} \qquad \gamma(R) = \begin{cases} 0 & \text{if } \kappa \geq 0; \\ -\kappa - 1 & \text{if } \kappa < 0. \end{cases}$$

Remark 2.26. $\beta(R)$ and $\gamma(R)$ are independent of B_1, B_2.

Proof. Let $v \in \ker R$. By Similarity principle (Theorem 2.12) we can write it in the form $v = \phi e^{\omega}$, where ϕ is holomorphic in $\mathbb{D}$ and ω is bounded. We obtain a holomorphic Riemann-Hilbert problem

$$\phi = \zeta^{\kappa} e^{\bar{\omega} - \omega}\overline{\phi}, \quad \zeta \in \partial\mathbb{D}$$

Note that for $\tau = e^{\bar{\omega}-\omega}$ is a non-vanishing function satisfying $\tau = \bar{\tau}^{-1}$ and that $\kappa = W(\zeta^{\kappa}\tau) = W(\zeta^{\kappa})$. By Corollary 2.9 we can deduce that $\beta(R) = 0$ if $\kappa = -2W(P) + 1 < 0$ from the scalar holomorphic theory. Further, $\gamma(R) = -\kappa - 1$ by the index formula.

We apply the part (2) of Proposition 2.24 to prove the rest. Note that $-P^T\overline{P}^{-T}$ admits a factorization $\Theta^{-T}\bar{\Lambda}\bar{\Theta}^T$ (expressed in terms of the above one). Further, $W(\overline{\frac{\partial\zeta}{\partial s}}/\frac{\partial\zeta}{\partial s}) = -2$. Hence κ_j is the partial index of R if and only if $-\kappa_j - 2$ is the partial index of the adjoint. □

We now follow these ideas to study the higher dimensional case. In contrast with the scalar case some of the indices may be odd. If none of the indices are odd, it is equivalent to consider $R_2(v) = 2\mathrm{Re}(\Lambda^{1/2}v)$

as the boundary condition of the simplified operator (2.25). Hence the scalar theory can be applied directly.

If the factorization admits some odd indices, they appear in pairs since the total index is even. In that case, the simplified condition with $\Lambda(\zeta) = \mathrm{diag}(\zeta^{2j-1}, \zeta^{2k-1})$ is equivalent to $R_2(v) = 2\mathrm{Re}(Kv)$ where

$$K(\zeta) = \begin{bmatrix} 1+\zeta & -i(1-\zeta) \\ i(1-\zeta) & 1+\zeta \end{bmatrix} \begin{bmatrix} \zeta^{-j} & 0 \\ 0 & \zeta^{-k} \end{bmatrix}$$

Thus, in general, the simplest form of $R_2(v) = \mathrm{Re}(Kv)$ has 2×2 blocks on the diagonal. This justifies the effort put into proving a higher dimensional index formula at the beginning of this section, although we could restrict to the above particular matrix K.

In what follows, we divide the indices in two sets. Let $1 \leq m \leq n$ be such that

$$\kappa_1 \geq \kappa_2 \geq \ldots \kappa_m \geq -1 > \kappa_{m+1} \geq \kappa_{m+2} \geq \ldots \geq \kappa_n.$$

The following computations hold in the holomorphic case $B_1, B_2 = 0$.

Proposition 2.27. *If $B_1, B_2 \equiv 0$ then*

$$\beta = \sum_{j=1}^{m} (\kappa_j + 1), \quad \gamma = \sum_{j=m+1}^{n} (-\kappa_j - 1).$$

Proof. Let $v \in \ker R$. Note that the boundary condition imposes some restrictions on the expansion of the entries

$$v_k = \sum_{j=1}^{n} c_{k,j} \zeta^j = \sum_{j=1}^{n} \bar{c}_{k,j} \zeta^{\kappa_k - j} = \zeta^{\kappa} \bar{v}_k \text{ for } \zeta \in \partial\mathbb{D}.$$

This allows to determine the dimension of the kernel explicitly. The rest follows from the index formula and the adjoint operator. □

In search of a generalized analytic result, it seems natural to mimic the proof of Proposition 2.25. A generalized analytic vector $v = S\phi$ can be represented globally by a nonsingular matrix S and a holomorphic vector ϕ (Remark 2.15). Since $W(\det S\Lambda\overline{S}^{-1}) = W(\det \Lambda)$, we obtain a holomorphic boundary problem of the same total index. However, this gives no information on the partial indices. Hence unlike in the scalar case, the dimensions $\beta(R)$ and $\gamma(R)$ depend on B_1 and B_2 in general. We illustrate this using the same example as in the previous section and computations from [**4**].

Example 2.28. Let $u = (u_1, u_2)$ be a general solution of

$$u_{\bar{\zeta}} + \begin{pmatrix} 0 & 6\zeta^2/(3-\zeta^2\bar{\zeta}^2) \\ -1 & 0 \end{pmatrix} u = 0.$$

We have shown in the previous section that

$$\begin{bmatrix} u_1(\zeta) \\ u_2(\zeta) \end{bmatrix} = \begin{bmatrix} \lambda_1\zeta\phi_1(\zeta) \\ \lambda_2\phi_1(\zeta) + \frac{2}{3}\bar{\zeta}\phi_2(\zeta) \end{bmatrix}, \ \zeta \in \partial\mathbb{D}. \tag{2.26}$$

Here ϕ_1, ϕ_2 are holomorphic and $\lambda_1, \lambda_2 \in \mathbb{R}$. We can consider two equivalent boundary problems:

$$\begin{array}{rcl} u_1 &=& \zeta^{\kappa_1}\overline{u_1} \\ u_2 &=& \zeta^{\kappa_2}\overline{u_2} \end{array} \iff \begin{array}{rcl} \phi_1 &=& \zeta^{\kappa_1-2}\overline{\phi_1} \\ \phi_2 &=& \zeta^{\kappa_2+2}\overline{\phi_2} + \frac{3}{2}\lambda_2\zeta(\zeta^{\kappa_2}\overline{\phi_1} - \phi_1) \end{array}.$$

Their Fredholm indices are equal to $\kappa_1 + \kappa_2 + 2$. Assume that $\kappa_2 \geq -3$. Given ϕ_1, the boundary problem for ϕ_2 is always solvable and admits a $\kappa_2 + 3$ parametric solution. Based on this observation, we obtain the following computations:

- if $\kappa_{1,2} = 0$ then $\beta = 3$, $\gamma = 1$
- if $\kappa_1 = 2$, $\kappa_2 = 0$ then $\beta = 4$, $\gamma = 0$
- if $\kappa_1 = 2$, $\kappa_2 = -2$ then $\beta = 2$, $\gamma = 0$.

In Proposition 2.27 the dimension $\gamma(R)$ was zero if and only if the indices were 'good', that is, greater or equal to -1. The first and the last case above yield contradiction for any similar statement in general.

It may seem that no singular integrals were used so far. This however, is not true. It is rather easy to compute the dimension $\beta(R)$ explicitly. Furthermore, by means of Green's formula some constraints in terms of the adjoint operator were provided. However, it is the sufficiency of these restrictions that was cleverly hidden into the Theorem 2.22. To fully answer the question, one has to construct the solutions explicitly. In order to do so, the holomorphic part of the Cauchy integral formula has to be studied.

We proceed by presenting an approach from Buchanan [**4**] based on the Cauchy-Green transform T from (2.10). For $u \in L^p(\mathbb{D})^n$, $p > 2$ we introduce the transform

$$T_*(u)(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{u(\xi)}{1-\bar{\xi}z}\,\mathrm{d}x\,\mathrm{d}y(\xi).$$

It arises from the outer part of T reflected over the unit circle. Since $zT_*(u)(z) = \overline{T(u)(1/\bar{z})}$, the function $v = T_*(u)$ is holomorphic on $\mathbb{D}$.

Assume that $\kappa_j \geq -1$ and that $R(u) = 0$. Then

$$\Psi(u)(z) = T(B_1 u + B_2 \bar{u})(z) + z\Lambda T_*(\overline{B}_1 \bar{u} + \overline{B}_2 u)(z) = \phi(z) \tag{2.27}$$

is a vector function holomorphic on $\mathbb{D}$ satisfying boundary condition $\phi = \Lambda\bar{\phi}$. The correspondence may have a nontrivial kernel when $n \geq 2$. Let us denote its real dimension by N.

Since $\ker(\Psi^*) = (\mathrm{range}(\Psi))^\perp$, the equation $\Psi(u) = \phi$ is solvable if and only if ϕ is orthogonal to the kernel of the adjoint integral operator defined with respect to the inner product (2.18). Since the Fredholm index equals $\nu(\Psi) = 0$, the kernel of the adjoint also has dimension N. Thus the maximal dimension of the restrictions equals the minimum of N and $\kappa + n$. However, some of them might be linearly dependent.

Once the admissible choice of ϕ is made, the solution can be found as follows. Suppose $\{u_j\}_{j=1}^N$ is an orthogonal basis for the space of solutions of the equation $\Psi(u) = 0$ and $\{v_j\}_{j=1}^N$ is a basis for the appropriate adjoint equation. Then the operator

$$\Psi_1(u) = \Psi(u) + \sum_{j=1}^{N} (u, u_j)\, v_j$$

is Fredholm with trivial kernel and thus invertible. For an admissible ϕ, the general solution of (2.27) is given by

$$u = \Psi_1^{-1}(\phi) + \sum_{j=1}^{N} c_j u_j$$

where c_j are arbitrary real constants. Hence $\beta(R) = \kappa + n + N - r$ for some $r \leq \min\{\kappa + n, N\}$.

Allowing the indices to be less than -1, the theory becomes complicated, hence we only cite the result here. As above one has to solve

$$\Psi(u) = T(B_1 u + B_2 \bar{u}) + \Lambda_1 T_* \overline{\Lambda}_2 (\overline{B}_1 \bar{u} + \overline{B}_2 u) = \phi, \tag{2.28}$$

for an admissible ϕ, where $\Lambda_1(\zeta) = \mathrm{diag}\,(\zeta^{\kappa_1+1}, \ldots, \zeta^{\kappa_m+1}, 1, \ldots, 1)$ and $\Lambda_2(\zeta) = \mathrm{diag}\,(1, \ldots, 1, \zeta^{\kappa_{m+1}+1}, \ldots, \zeta^{\kappa_n+1})$. However, a solution u does not necessarily solve the initial boundary problem. Some solvability conditions are to be fulfilled (they may not be independent of the ones imposed when seeking an admissible ϕ). In [**4**, p. 135] Buchanan gives explicit method for calculating this defect r as the rank of a $\left(\sum_{j=1}^m (\kappa_j + 1) + N\right) \times \left(\sum_{j=m+1}^n (-\kappa_j - 1) + N\right)$ real matrix.

COROLLARY 2.29.

$$\beta = \sum_{j=1}^{m}(\kappa_j + 1) + N - r, \quad \gamma = \sum_{j=m+1}^{n}(-\kappa_j - 1) + N - r.$$

The rank r is expressing the relation of these coefficients B_1, B_2 with the boundary matrix Λ. In the holomorphic case, we have $N = r = 0$, the same is true for $n = 1$. In the Example 2.28, we have $N = 2$ and:

- if $\kappa_{1,2} = 0$ then $r = 1$, $\beta = 3$, $\gamma = 1$
- if $\kappa_1 = 2$, $\kappa_2 = 0$ then $r = 2$, $\beta = 4$, $\gamma = 0$
- if $\kappa_1 = 2$, $\kappa_2 = -2$ then $r = 3$, $\beta = 2$, $\gamma = 0$.

CHAPTER 3

Deformations of J-holomorphic discs

In this chapter we return to the theory of pseudoholomorphic discs. We recall the J-holomorphicity condition for $u\colon \mathbb{D} \to \mathbb{C}^n$ presented in the first chapter. Assuming that $\det(J(u) + J_{st}) \neq 0$, we have

$$u_{\bar\zeta} + A(u)\overline{u_\zeta} = 0, \tag{3.1}$$

where $A(u)(v) = (J_{st} + J(u))^{-1}(J(u) - J_{st})(\bar v)$ is a complex $n \times n$ matrix function. After having introduced the Cauchy-Green transform in (2.6), we can rewrite (3.1) into

$$\Phi^J(u) = u + T\left[A(u)\overline{u_\zeta}\right] = \phi, \tag{3.2}$$

where ϕ is a usual holomorphic vector function. Let $k \in \mathbb{N}_0$. Then the operator Φ^J maps the space $W^{k,p}(\mathbb{D})$, $p > 2$, and the space $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})^n$, $0 < \alpha < 1$, to itself. If $J \in \mathcal{C}^k$, a $W^{1,p}(\mathbb{D}) \cap \mathcal{C}(\mathbb{D})$-regular disc is in $C^{k,\alpha}(\mathbb{D})$ for every $0 < \alpha < 1$ [**20**]. We restrict to the discs that are regular up to the boundary. Precisely, we assume them to be J-holomorphic on some neighborhood of $\overline{\mathbb{D}}$ (this is meant when we say that $u\colon \overline{\mathbb{D}} \to M$ is J-holomorphic).

We study invertibility of Φ^J in order to establish a one-to-one correspondence between the J-holomorphic discs and the usual holomorphic discs in $\mathbb{C}^n$. We add a certain boundary condition in § 3.2. The proofs use in an important way the local and linear theory developed in the previous two chapters. Furthermore, two geometric applications are obtained in § 3.3. We consider them as a preparation for § 3.4, where an application from the theory of disc functionals is given.

3.1. Neighborhood of a J-holomorphic discs

We prove that, given an immersed J-holomorphic disc, one can always perturb it holomorphically. Our approach is based on the Inverse function theorem. The proofs are short and simple since the main work was done in the first two chapters. We suggest the reader to consult the sections § 1.4, § 1.5 and § 2.3 beforehand.

We begin by proving the classical Nijenhuis-Woolf theorem we have already used in the first chapter. We give an extended version from [**19**] interpolating the higher order jets (see also [**6**]).

Suppose that $k \in \mathbb{N}$ and let $\{\varphi_\eta\}_\eta$ be a set of compatible charts mapping open subsets U_η of M to the space $\mathbb{R}^{2n}$. For $p \in U_\eta$ we think of the k-th jet space $\mathcal{J}_p^k M$ as the set of equivalence classes $[\eta, V]$ with $V = (v_1, v_2, \dots, v_k) \in (\mathbb{R}^{2n})^k$ under the relation: $[\eta, V] \sim [\eta', V']$ if and only if there exist a curve $\gamma\colon [-1,1] \to M$ such that $\gamma(0) = p$ and

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j}|_{t=0} (\varphi_\eta \circ \gamma) = v_j \text{ and } \frac{\mathrm{d}^j}{\mathrm{d}t^j}|_{t=0} (\varphi_{\eta'} \circ \gamma) = v_j', \quad j = 1, 2, \dots, k.$$

Note that $\mathcal{J}_p^1 M \cong T_p M$ (see § 1.1). Let $\zeta = x + iy \in \mathbb{D}$. We say that the k-th jet of a disc u at the origin is equal to $[\eta, V] \in \mathcal{J}_p^k M$ if

$$\frac{\partial^j}{\partial x^j} (\varphi_\eta \circ u)(0) = v_j, \quad j = 1, 2, \dots, k.$$

Recall that for pseudoholomorphic discs in $\mathbb{R}^{2n}$ one has (1.2). Hence in this case, the above condition determines all the local derivatives of u up to the k-th order.

THEOREM 3.1. *Let $k \in \mathbb{N}$. Let M be an almost complex manifold equipped with a $\mathcal{C}^k$-smooth structure J. For every $p \in M$ and $[\eta, V] \in \mathcal{J}_p^k M$ there exist $\lambda > 0$ and a J-holomorphic disc $u_{p,V}$ such that $u_{p,V}(0) = p$ and the k-th jet of $u_{p,V}$ at the origin is equal to $[\eta, \lambda V]$.*

PROOF. By Lemma 1.17 there exists a chart φ satisfying $\varphi(p) = 0$ and $\varphi_*(J)(p) = J_{st}$. Hence it suffices to prove that for every almost complex structure J defined on a neighborhood of the origin with property $J(0) = J_{st}$ and every small $V = (v_1, v_2, \dots, v_k) \in (\mathbb{R}^{2n})^k$ there exist a J-holomorphic disc $u_{0,V}$ in $\mathbb{R}^{2n}$ such that $u_{0,V}(0) = 0$ and

$$\frac{\partial^j u_{0,V}}{\partial^j \zeta^j}(0) = \frac{\partial u_{0,V}}{\partial x^j}(0) = v_j, \ j = 1, 2, \dots, k.$$

Note that the two derivatives agree since $J(0) = J_{st}$.

Furthermore, we can assume that the $\mathcal{C}^1$-norm of the complex matrix A is small. For convenience, we use the normalized Cauchy-Green transform from (2.22) and define Φ_0^J similarly to (3.2). The operator is a small perturbation of identity mapping on $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$ and hence invertible. That is, the equation $\Phi_0^J(u) = \phi$ has a unique solution $u = u(\phi)$ for every small enough $\phi \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$. Note that also $u(0) = \phi(0)$.

We define

$$\phi_{0,W}(\zeta) = \sum_{l=1}^{k} \frac{1}{l!} w_l \zeta^l$$

for $W = (w_1, w_2, \ldots, w_k) \in (\mathbb{R}^{2n})^k$. For small W we can define the J-holomorphic disc $u_{0,W} = (\Phi_0^J)^{-1}(\phi_{0,W})$. Furthermore, the linear map

$$W \mapsto \left(\frac{\partial u_{0,W}}{\partial \zeta}(0), \ldots, \frac{\partial^k u_{q,W}}{\partial \zeta^k}(0) \right)$$

is a small continuous perturbation of the identity. Hence there exists a $W \in (\mathbb{R}^{2n})^k$ that is mapped into V. □

Remark 3.2. We gave no proof here on the smooth dependence of the disc $u_{p,V}$ on (p, V) and J in the Nijenhuis-Woolf theorem. One should check [**35, 6**]. In [**19**] the above theorem is proved for J only Hölder continuous by using Schauder fixed point theorem.

In the same spirit, we study Φ^J on a neighborhood of an embedded J-holomorphic disc. In contrast, the derivatives of A can not be considered to be small in general. Hence a different argument is needed. We start with a special case of an embedded disc in a four dimensional real manifold. The low dimension will enable us to use the theory of generalized analytic functions discussed in the previous chapter.

Proposition 3.3. *Let $k \in \mathbb{N}$ and $0 < \alpha < 1$. Let (M, J) be an almost complex manifold with $J \in \mathcal{C}^k$ and $\dim_{\mathbb{R}} M = 4$. There exists a one-to-one correspondence between the space of all J-holomorphic discs in a small $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$-neighborhood of a given embedded J-holomorphic disc and the usual holomorphic disc in $\mathbb{C}^2$.*

Proof. Recall that by the Corollary 1.21 we can restrict our consideration to a flat disc $u_0(\zeta) = (0, \zeta)$ in $\mathbb{C}^2$, equipped with a structure that is standard along its image. Further, let us think of the case when A equals A_1 from Proposition 1.22 near $u_0(\overline{\mathbb{D}})$. Since J_1 is assumed to be $\mathcal{C}^1$ at least, the operator Φ^{J_1} is a continuously differentiable map. Its Fréchet derivative at the point u_0 is the map

$$\mathrm{D}\Phi^{J_1}(u_0)\colon (v_1, v_2) \mapsto (v_1 + T(B_1 v_1 + B_2 \overline{v_1}), v_2).$$

By Proposition 2.13 the derivative $\mathrm{D}\Phi^{J_1}(u_0)$ is an isomorphism and thus, by the Inverse function theorem, Φ^{J_1} is invertible on a neighborhood of u_0. Furthermore, by Proposition 1.22 the operator Φ^J is close to Φ^{J_1} and thus invertible on a neighborhood of u_0. □

The above arguments can not be used in higher dimensions. Although one can reproduce the result of Proposition 1.22 (Remark 1.26) the linearization of Φ^J may have a nontrivial kernel (Example 2.16). However, one can modify the correspondence by adding a linear term similar to the one from Theorem 2.18. This implies a general result from Sukhov and Tumanov [**36**]. Moreover, since no dimension restrictions is needed, one can consider the graph of a given disc.

Let $u\colon \overline{\mathbb{D}} \to M$ be a J-holomorphic disc. Its graph is a pseudo-holomorphic embedding mapping the closed unit disc to the manifold $(M \times \mathbb{R}^2, J \oplus J_{st})$. We apply the procedure from the Corollary 1.21 and find a coordinate map $\Sigma(q,\zeta) = (z(q,\zeta),\zeta)$ defined in the neighborhood of the graph such that the push forward $\tilde{J} = \Sigma_*(J \oplus J_{st})$ defines an almost complex structure in a neighborhood of $\{0\} \times \overline{\mathbb{D}}$ and $\tilde{J}(0,\zeta) = J_{st}$. Moreover, since the projection $(z,\zeta) \to \zeta$ is $(\tilde{J}, J_{st})$-holomorphic by (1.6), the complex matrix $\tilde{A}$ of $\tilde{J}$ is of the form

$$\tilde{A} = \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix}.$$

A disc v close to u is J-holomorphic if and only if its graph is $J \oplus J_{st}$-holomorphic. The latter is equivalent to $(z(\zeta, v(\zeta)),\zeta)$ being $\tilde{J}$-holomorphic, that is, if and only if $h(\zeta) = z(v(\zeta),\zeta)$ satisfies

$$h_{\bar{\zeta}} + A(h,\zeta)\overline{h_\zeta} + b(h,\zeta) = 0.$$

We seek solutions close to the origin. Note that the above equation is more general than the usual J-holomorphicity condition (3.1) since it is non-homogeneous and A varies with ζ. We define

$$\Upsilon^J(h) = h + T_0\left(A(\zeta,h)\overline{h_\zeta} + b(\zeta,h)\right).$$

We again use the normalized Cauchy-Green transform T_0 from (2.22). The Fréchet derivative of Υ^J at the zero constant disc is of the form

$$\mathrm{D}\Upsilon^J(0)(h) = h + T_0(B_1 h + B_2 \bar{h}),$$

where the matrices B_1 and B_2 arise from A and b. Note that there is no $\overline{h_\zeta}$ term since $A(\zeta,0) = 0$. The linear operator is Fredholm, but not necessarily invertible. We construct a linear holomorphic modification from the Remark 2.21 in order to provide bijectivity. We add such a term to the original correspondence:

$$\tilde{\Upsilon}^J(h) = \Upsilon^J(h) + \sum_{j=1}^{N} \mathrm{Re}(h, w_j) p_j^0.$$

Hence small solutions are identified with the holomorphic discs in $\mathbb{C}^n$. This is crucial for the theorem from [**36**] cited below.

THEOREM 3.4. *Let $k \in \mathbb{N}$ and $0 < \alpha < 1$. Let (M, J) be an almost complex manifold with $J \in \mathcal{C}^\infty$. The set of J-holomorphic discs of class $\mathcal{C}^{k,\alpha}(\mathbb{D})$ forms a smooth Banach manifold modeled on the space of holomorphic functions $\mathbb{D} \to \mathbb{C}^n$ of the same class.*

We point out that Forstnerič [**13**] proved an analogous result earlier for the usual complex structure and holomorphic mappings of a strongly pseudoconvex domain in a Stein manifold.

3.2. Discs attached to maximal totally real manifolds

In this section, we study the structure of the space of J-holomorphic discs near an embedded J-holomorphic disc attached along their boundaries to a maximal totally real manifold E of $(\mathbb{C}^n, J)$. We point out that in the two dimensional complex space the space of all J-holomorphic deformations of the disc does not depend on the almost complex structure in a neighborhood of the disc, but when $n > 2$ this in not true. Such an observation is an original result from [**21**]. We give sufficient conditions, in terms of an elliptic boundary value problem from § 2.4 for the set of all nearby attached discs to form a manifold.

A $\mathcal{C}^1$ submanifold E of codimension n in $(\mathbb{R}^{2n}, J)$ is called *maximal totally real* if the maximal J-complex subspace of the tangent space T_pE is trivial, that is, if $T_pE \cap J(p)T_pE = \{0\}$ holds for all $p \in E$. A disc u is said to be *attached to a submanifold* E of $\mathbb{R}^{2n}$ if u maps the boundary circle $\partial\,\mathbb{D}$ into E.

For convenience of notation we work in the spaces $\mathbb{R}^{2n+2}$ and $\mathbb{C}^{n+1}$, respectively, using the notation $\tilde{v} = (v, v_{n+1}) \in \mathbb{C}^n \times \mathbb{C}$. Suppose that $u\colon \overline{\mathbb{D}} \to \mathbb{R}^{2n+2}$ is a J-holomorphic embedding with boundary attached a to maximal real submanifold E. By Corollary 1.21 one can find a diffeomorphism in a neighborhood of $u(\overline{\mathbb{D}})$ such that locally the disc equals to $u_0(\zeta) = (0, 0, \ldots, 0, \zeta)$ and the structure is standard along the image $u_0(\overline{\mathbb{D}})$. The totally real submanifold E is given locally near the circle $\{u(e^{i\theta})\colon \theta \in \mathbb{R}\}$ as the zero set of a smooth $\mathbb{R}^{n+1}$-valued defining function $\rho = (\rho_1, \rho_2, \ldots, \rho_{n+1})$ with $\partial\rho_1 \wedge \partial\rho_2 \wedge \ldots \wedge \partial\rho_{n+1} \neq 0$ (this means that the complex gradients of the component functions ρ_j are $\mathbb{C}$-linearly independent, which is equivalent to $E = \{\rho = 0\}$ being totally real). Since the disc was reduced to $\{0\}^n \times \overline{\mathbb{D}}$, the defining function can be chosen such that for z_{n+1} near $\partial\mathbb{D}$

$$\rho(0, z_{n+1}) = \big(0, \ldots, 0, |z_{n+1}|^2 - 1\big).$$

We can choose a small enough neighborhood U of the disc u_0 in $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})^{n+1}$ so that any $u \in U$ is attached to E if and only if $\rho \circ u = 0$ on $\partial\mathbb{D}$. Consider the mapping

$$R^J = (R^J_1, R^J_2) \colon \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})^{n+1} \to \mathcal{C}^{k-1,\alpha}(\overline{\mathbb{D}})^{n+1} \times \mathcal{C}^{k,\alpha}_{\mathbb{R}}(\partial\mathbb{D})^{n+1}$$

given by

$$R^J_1(u) = u_{\bar\zeta} + A(u)\overline{u_\zeta} \;\text{ on } \overline{\mathbb{D}}, \quad R^J_2(u) = \rho \circ u \;\text{ on } \partial\mathbb{D}.$$

Its zero set

$$\mathcal{M} = \{u \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})^{n+1} \colon u \text{ near } u_0,\; R^J(u) = 0\}$$

is precisely the set of all J-holomorphic discs close to u_0 and attached to E. Our aim is to give sufficient conditions for this subset of $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})^{n+1}$ to be a manifold. By Corollary 2.5, it suffices to prove that the Fréchet derivative $DR^J(u_0)$ of the map R^J, calculated at the disc u_0, is surjective, and that its kernel is complemented in $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})^{n+1}$.

Using the above normalization, one easily finds that

$$DR^J_1(u_0)(\tilde v) = \tilde v_{\bar\zeta} + \tilde B_1 \tilde v + \tilde B_2 \overline{\tilde v} \;\text{ on } \overline{\mathbb{D}}, \quad DR^J_2(u_0)(\tilde v) = 2\mathrm{Re}(\tilde P \tilde v) \text{ on } \partial\mathbb{D}.$$

Here we have

$$\widetilde{B}_1 = \begin{pmatrix} B_1 & 0 \\ b_1 & 0 \end{pmatrix}, \quad \widetilde{B}_2 = \begin{pmatrix} B_2 & 0 \\ b_2 & 0 \end{pmatrix}, \quad \widetilde{P} = \begin{pmatrix} P & 0 \\ p & \bar\zeta \end{pmatrix},$$

where

$$B_1(\zeta) = \left(\frac{\partial a_{j,n+1}}{\partial z_k}(0,\zeta)\right)_{j,k=1}^{n}, \qquad b_1(\zeta) = \left(\frac{\partial a_{n+1,n+1}}{\partial z_k}(0,\zeta)\right)_{k=1}^{n},$$

$$B_2(\zeta) = \left(\frac{\partial a_{j,n+1}}{\partial \bar z_k}(0,\zeta)\right)_{j,k=1}^{n}, \qquad b_2(\zeta) = \left(\frac{\partial a_{n+1,n+1}}{\partial \bar z_k}(0,\zeta)\right)_{k=1}^{n},$$

$$P(\zeta) = \left(\frac{\partial \rho_j}{\partial z_k}(0,\zeta)\right)_{j,k=1}^{n}, \qquad p(\zeta) = \left(\frac{\partial \rho_{n+1}}{\partial z_k}(0,\zeta)\right)_{k=1}^{n},$$

and $a_{i,j}$ are the entries of the complex matrix A. In particular, we have $\widetilde{B}_1 \equiv 0$, $\widetilde{B}_2 \equiv 0$ in the standard case $J \equiv J_{st}$.

One has to study the following two systems:

$$\begin{cases} v_{\bar\zeta} + B_1 v + B_2 \overline{v} = \psi & \text{on } \overline{\mathbb{D}}, \\ \mathrm{Re}(Pv) = \phi & \text{on } \partial\mathbb{D}, \end{cases} \tag{3.3}$$

and

$$\begin{cases} (v_{n+1})_{\bar\zeta} = \psi_{n+1} - b_1 v - b_2 \overline{v} & \text{on } \overline{\mathbb{D}}, \\ \mathrm{Re}(\bar\zeta v_{n+1}) = \phi_{n+1} - pv & \text{on } \partial\mathbb{D}. \end{cases}$$

Assuming for a moment that we already have a solution to the first boundary value problem (3.3), the second one is just an ordinary holomorphic Riemann-Hilbert problem whose general solution depends on three real parameters (Theorem 2.22). So the question is reduced to a Riemann-Hilbert problem for the Pascali type system (3.3), that is, the operator R defined in (2.23). We state the sufficient conditions in terms of $\beta(R)$ and $\gamma(R)$ defined as in (2.24).

Example 3.5. Suppose $n = 1$. By Proposition 2.25 the sufficient condition for $\gamma(R) = 0$ is $\kappa \geq 0$ and $\dim_{\mathbb{R}} \mathcal{M} = \kappa + 4$. This result was given by Forstnerič [**12**] in the standard case, but in terms of $W(P)$. He also pointed out how in case of a 'bad' index $\mathcal{M}$ only consists of a three parametric family of unit disc automorphisms. In [**18**] this result was obtained for any structure in terms of so-called *Maslov index*. As pointed out (Proposition 2.25), the sufficient condition does not depend on B_1 and B_2, that is, in dimension two the question of $\mathcal{M}$ being a manifold is invariant of the almost complex structure J.

Example 3.6. Suppose $n > 1$ but $J \equiv J_{st}$. By Proposition 2.27 we have $\gamma(R) = 0$ if and only if $\kappa_j \geq -1$ for every $j = 1, 2, \ldots, n$. In this case, the real dimension of $\mathcal{M}$ equals $\kappa + n + 3$. This particular result was given by Globevnik in [**14**] (see also the paper of Oh [**29**]). In comparison with his result, we have fixed the index in the tangential direction to be $\kappa_{n+1} = 2$ when choosing an embedded disc.

In general, the dimension $\gamma(R)$ depends on the matrices B_1 and B_2 (Example 2.28), and thus the problem depends on the almost complex structure J. Precisely, it depends on the linear terms in the last column of the matrix A. By Proposition 1.12 this dependence can be stated in terms of the linear term of the last column in the original structure. Recall the transformation rule for matrix the A under diffeomorphisms (Lemma 1.15). We see that a holomorphic change of coordinates fixing the disc u_0 preserves the property $A = 0$. Hence one may find a normal coordinate form for embedded J-holomorphic disc attached to E in which u equals to u_0, $J(u) = J_{st}$ and $-P^{-1}\bar{P} = \Lambda$, where the matrix functions P and Λ are defined as in (2.25).

In this form, we discuss the condition using the Corollary 2.29:

$$\sum_{j=m+1}^{n} (-\kappa_j - 1) + N = r.$$

The task of describing all cases when this condition is fulfilled is far from trivial. We can get generalization of the condition in the standard case when all partial indices are ≥ -1 and the integral operator Ψ defined as in (2.27) has a trivial kernel. This happens in particular, when B_1 and B_2 are diagonal. However, in general we may have 'good' indices with $\gamma(R) > 0$ or reach the sufficient conditions even in the case of a 'bad' index (Example 2.28).

Remark 3.7. In the unpublished preprint of J.M.Trepreau another type of sufficient conditions for the standard theory was given. The condition was based on the characterization of the range of R with real functions, but also had a geometric interpretation. The equivalence of Trepreau type conditions with the ones given in Globevnik's paper was proved by Černe in [**8**]. The question whether an analogous result can be obtained for the non-integrable has not been studied yet.

3.3. Two applications

This section contains two unrelated results applying the work done in § 3.1. Firstly, we approximate a disc that is close to being pseudo-holomorphic by a J-holomorphic one. We deal with spaces $\mathbb{R}^{2n}$ only to avoid obstructions arising when solving such a problem in general. Still the result is new [**23**] and might be an initial step in obtaining a stronger result.

Secondly, we present a result from [**7**] providing a disc attached to a given real torus. The tori and the discs under our consideration will not be arbitrary. Given a immersed pseudoholomorphic disc u, we associate to u a real 2-dimensional torus formed by the boundary circles of discs centered at the boundary points $u(\zeta)$, $\zeta \in \partial\mathbb{D}$. Furthermore, the method requires restriction to the manifolds of real dimension four. However, such a geometric construction will play a crucial role when proving the main theorem in § 3.4.

3.3.1. Approximation of non-holomorphic discs. We are interested in the general question: Is every map from the unit disc into a complex manifold, with small $\bar{\partial}$, close to a holomorphic map in $L^\infty(\mathbb{D})$. Introducing the transform T in (2.6) yields a simple answer in $\mathbb{C}^n$. Let h be a disc in $\mathbb{C}^n$ with $\|h\|_{L^\infty(\mathbb{D})} < \delta$. Then the disc

$$u = h - Th$$

is holomorphic. Further, $\|u - h\|_{L^\infty(\mathbb{D})} \leq C\delta$ for some $C > 0$ by Proposition 2.10. One can even fix the center using the normalization (2.22). We generalize this in the following proposition.

PROPOSITION 3.8. *Let $h\colon \overline{\mathbb{D}} \to \mathbb{R}^{2n}$ be smooth. Let J be a smooth almost complex structure defined in a neighborhood of the image $h(\overline{\mathbb{D}})$ such that* $\det(J + J_{st}) \neq 0$. *Define A to be the complex matrix of J. For every $\epsilon > 0$ one can find a $\delta > 0$ such that if*

$$\left\|h_{\bar{\zeta}} + A(h)\overline{h_\zeta}\right\|_{L^\infty(\mathbb{D})} < \delta$$

we have a J-holomorphic disc u with $u(0) = h(0)$ and $\|h - u\|_{L^\infty(\mathbb{D})} < \epsilon$.

PROOF. We introduce a substitution by linear transformation

$$v = u + A(h)\bar{u} \quad \Longleftrightarrow \quad u = (I - A(h)\overline{A(h)})^{-1}(v - A(h)\bar{v}).$$

Close to $h(\overline{\mathbb{D}})$ we rewrite the condition (1.4) in the form

$$v_{\bar{\zeta}} + K_0(v)\overline{v_\zeta} + K_1(v)v + K_2(v)\bar{v} = 0, \tag{3.4}$$

where K_j $(j = 0, 1, 2)$ are matrices arising from A. Set $\psi = h + A(h)\overline{h}$. Suppose that

$$\left\|h_{\bar{\zeta}} + A(h)\overline{h_\zeta}\right\|_{L^\infty(\mathbb{D})} = \left\|\psi_{\bar{\zeta}} + K_1(\psi)\psi + K_2(\psi)\bar{\psi}\right\|_{L^\infty(\mathbb{D})} < \delta.$$

It suffices to show that for any $\epsilon > 0$ one can fix $\delta > 0$ so that there exists a v solving (3.4) such that $v(0) = \psi(0)$ and $\|\psi - v\|_{L^\infty(\mathbb{D})} < \epsilon$.

Using the normalized transform T_0 from (2.22) we define

$$\Gamma(v) = v + T_0(K_0\overline{v_\zeta} + K_1 v + K_2\bar{v})$$

mapping the space $W^{1,p}(\mathbb{D})$ smoothly to itself. Its Fréchet derivative at the point ψ is of the form

$$D\Gamma(\psi)(w) = w + T_0(B_1 w + B_2\bar{w}),$$

where the matrices B_1 and B_2 arise from the matrices K_j, $j = 0, 1, 2$. Note that there is no $\overline{w_\zeta}$ term since $K_0(\psi) = 0$. The derivative is Fredholm but not necessarily invertible (Example 2.16).

In the spirit of Theorem 2.18 and Remark 2.21 from the second chapter we modify the linear operator adding a holomorphic term to obtain bijectivity. We further use this term to modify Γ :

$$\tilde{\Gamma}(v) = \Gamma(v) + \sum_{j=1}^{N} \mathrm{Re}(v, w_j) p_j^0.$$

By the implicit function theorem $\tilde{\Gamma}^{-1}$ is well defined and smooth in a $W^{1,p}(\mathbb{D})$-neighborhood of $\tilde{\Gamma}(\psi)$. We stress that $\tilde{\Gamma}(v)(0) = v(0)$.

Define $w = \left(\tilde{\Gamma}(\psi)\right)_{\bar{\zeta}} = \psi_{\bar{\zeta}} + K_1(\psi)\psi + K_2(\psi)\bar{\psi}$ and assume that $|w| < \delta$. For δ small enough $\|T_0(w)\|_{W^{1,p}(\mathbb{D})}$ is small . One can define

$$v = \tilde{\Gamma}^{-1}\left(\tilde{\Gamma}(\psi) - T_0(w)\right).$$

By the construction v is a solution to (3.4) with $v(0) = \psi(0)$. Furthermore, it is close to ψ in the space $W^{1,p}(\mathbb{D})$. The rest follows since the inclusion $W^{1,p}(\mathbb{D}) \subset \mathcal{C}^{\alpha}(\mathbb{D})$ is bounded (Theorem 2.8). □

Remark 3.9. Assuming that $\det(J + J_{st}) \neq 0$, the smallness of the expression $\left\|h_{\bar{\zeta}} + A(h)\overline{h_{\zeta}}\right\|_{L^{\infty}(\mathbb{D})}$ is equivalent to $\left|\bar{\partial}_J \varphi\right|$ being small (with respect to some Riemann metric). However, the proposition can not be trivially extended to almost complex manifolds. The difficulties may arise even in the integrable case. In [**32**] the author gives solutions in the particular case of maps whose image is contained in a finite number of local charts. He obtains them applying the Cartan lemma with bounds and solving the Cousin problem. We believe that it might be possible to mimic the proof in the almost complex setting.

3.3.2. Attaching discs to a real torus. We present here a geometric construction from [**7**] needed in the next section. Given a real torus we attach a disc to its boundary. We restrict ourselves to almost complex manifolds (M, J) with J smooth and $\dim_{\mathbb{R}} M = 4$.

Formulation of the problem: Let u be a J-holomorphic immersion defined on $\mathbb{D}_\gamma := (1+\gamma)\mathbb{D}$ for $\gamma > 0$. Given a neighborhood U of $\partial\mathbb{D}$, we consider J-holomorphic discs $v_z \colon \mathbb{D}_\gamma \to M$, $z = x + iy \in U$, satisfying the condition $v_z(0) = u(z)$ and such that the direction $\mathrm{d}u_z\left(\frac{\partial}{\partial x}\right)$ is not tangent to u. Furthermore we assume that the map

$$G \colon U \times \mathbb{D}_\gamma \to M, \quad G(z, \zeta) = v_z(\zeta)$$

is smooth and locally diffeomorphic. We seek a J-holomorphic disc centered at $u(0)$ and attached to the real torus $\Lambda = G(\partial\mathbb{D} \times \partial\mathbb{D})$.

Step one: Extending the family G.
We would like to apply the Nijenhuis-Woolf theorem (Theorem 1.13) and construct small discs with centers in the set $\mathbb{D}\backslash U$ that will smoothly extend the family G. We denote by $X_z := du_z\left(\frac{\partial}{\partial x}\right)$ the vector field defined along $u(U)$. In general it is impossible to extend X as a non-vanishing vector field transversal to $u(\overline{\mathbb{D}})$ at every point. A reparametrization of the boundary discs might be needed. However, the set Λ remains the same.

We fix $m \in \mathbb{Z}$ and consider the discs $v_z^m(\zeta) := v_z(z^m\zeta)$, $\zeta \in \overline{\mathbb{D}}$. Their tangent vectors at the point $u(z)$ are equal to $X_z^m = (x + yJ)^m X_z$. We claim that, after a suitable choice of m, the vector field X_z^m can be extended in the desired way. Indeed, fix an arbitrary vector field Y transversal to $u(\overline{\mathbb{D}})$ at every point. By Nijenhuis-Woolf theorem we have a family of J-holomorphic discs $h_z \colon \overline{\mathbb{D}} \to M$ tangent to Y and such that $h_z(0) = u(z)$. We may assume the map $H(z,\zeta) := h_z(\zeta)$ to be a local diffeomorphism mapping a neighborhood of the closed bidisc $\overline{\mathbb{D}}^2$ onto a neighborhood of $u(\overline{\mathbb{D}})$ (see the proof of Theorem 3.1). Hence we can use the coordinates $(z,w) \in \overline{\mathbb{D}}^2$ and pull back the vector field X by H^{-1}. Let k denote the winding number of the w component of the vector field $z \to (H^{-1})_*(X_z)$ when the component z runs around the boundary circle of $(z,0)$. We set $m = -k$. The vector field $(H^{-1})_*(X^m)$ extends smoothly to the disc $z \mapsto (z,0)$, $z \in \overline{\mathbb{D}}$, as a vector field Z transversal to this disc at every point. Thus $\tilde{X} = G_*(Z)$ is the desired extension. We now replace the initial family G by $G^n(z,\zeta) := v_z^n(\zeta)$. For simplicity we do not change the notation.

Furthermore, fix $z \in \mathbb{D}$ and a small neighborhood of $u(z)$ such that the set of small J-holomorphic discs near $u(z)$ is in bijective correspondence with the set of small standard holomorphic discs. Let V_z be the set of all $z' \in \mathbb{D}$ for which $u(z')$ lies in this neighborhood. We cover the boundary of $\mathbb{D}\backslash U$ with a finite number of such sets $V_{z_1}, V_{z_2}, \ldots, V_{z_r}$. Let $V = \cup_{j=1}^r V_{z_j}$. We define a smooth cut-off function χ on $U \cap \mathbb{D}$ such that it equals to 1 on a slightly smaller closed subset of $(U \cap \overline{\mathbb{D}})\backslash V$ and that it has a small positive value on a slightly smaller closed subset of $U \cap V$. We redefine the discs v_z into the maps $\zeta \to v_z(\chi(z)\zeta)$. The disc with centers in $U \cap V$ are now small, but still tangent to $\tilde{X}$. We construct small discs with this same property and centers in $\mathbb{D}\backslash U$. The construction may be done in such a way that the family G, now extended to the whole bidisc, is locally diffeomorphic.

Step two: Finding the complex matrix on the bidisc.
We define $\tilde{J} = \mathrm{d}G^{-1} \circ (J) \circ \mathrm{d}G$ on a neighborhood of the closed bidisc. In the spirit of (3.1), we would like to find a complex $\tilde{J}$-holomorphicity condition. Every fiber $G(z, \,.\,)$ is J-holomorphic, hence given $z \in \overline{\mathbb{D}}$ the disc $w \mapsto (z, w)$, $w \in \overline{\mathbb{D}}$, is $\tilde{J}$-holomorphic. By (1.2) the matrix $\tilde{J}$ is of the 2×2 block form

$$\tilde{J} = \begin{bmatrix} J_{11} & 0 \\ J_{21} & J_{st} \end{bmatrix}.$$

We have to prove that $J_{11} + J_{st}$ (and thus $\tilde{J} + J_{st}$) is invertible. Since the disc $z \mapsto (z, 0)$, $z \in \overline{\mathbb{D}}$ is $\tilde{J}$-holomorphic, one has $\tilde{J}(z, 0) = J_{st}$. By Example 1.3 we have real functions $c(z, w)$ and $d(z, w) > 0$ such that

$$\det(J_{11}(z, w) + J_{st}) = 2c(z, w)^2 + 1 + \frac{c(z, w)^2 + 1}{d(z, w)} > 0.$$

Hence the discs $\zeta \mapsto (z(\zeta), w(\zeta))$ are $\tilde{J}$-holomorphic if and only if

$$z_{\bar{\zeta}} = a(z, w)\overline{z_\zeta}, \qquad w_{\bar{\zeta}} = b(z, w)\overline{z_\zeta} \tag{3.5}$$

for some $a, b : \mathbb{D}_\gamma \times \mathbb{D}_\gamma \to \mathbb{C}$ smooth, $a(z, 0) = b(z, 0) = 0$ and

$$|a(z, w)| = \left|\frac{\det(J_{11} - J_{st})}{\det(J_{11} + J_{st})}\right| < q < 1.$$

Step three: Solving the boundary problem.
The construction of the disc h attached to Λ is now reduced to finding a solution of the above system attached to the boundary of the bidisc $|z(\zeta)| = |w(\zeta)| = 1$ for $\zeta \in \partial\mathbb{D}$ that stays inside its small neighborhood.

LEMMA 3.10. *For $c \in \partial\mathbb{D}$ and $n \in \mathbb{N}$ there exists a function pair $(u, v) = (u_{c,n}, v_{c,n})$, such that the disc $(z(\zeta), w(\zeta)) = (\zeta e^{u(\zeta)}, c\zeta^n e^{v(\zeta)})$ is $\tilde{J}$-holomorphic and attached to $\partial\mathbb{D} \times \partial\mathbb{D}$. Furthermore, $\|w\|_\infty < 1 + \gamma$ for n big enough and $\|u_{c,n}\|_\infty \to 0$, when $n \to \infty$.*

PROOF. We seek a solution of (3.5) in the form

$$z(\zeta) = \zeta e^{u(\zeta)}, \; w(\zeta) = c\zeta^n e^{v(\zeta)}.$$

We have the following boundary value problem for u and v

$$\begin{aligned} u_{\bar{\zeta}} &= A(u, v, \zeta)(1 + \overline{\zeta u_\zeta}) \text{ on } \mathbb{D}, \\ v_{\bar{\zeta}} &= B(u, v, \zeta)(1 + \overline{\zeta u_\zeta}) \text{ on } \mathbb{D}, \\ \mathrm{Re}(u) &= \mathrm{Re}(v) = 0 \text{ on } \partial\mathbb{D}. \end{aligned}$$

Here $A(u,v,\zeta) = ae^{\bar{u}-u}/\zeta$ and $B(u,v,\zeta) = be^{\bar{u}-v}/\zeta^n$. Let us insert $h = u_{\bar{\zeta}}$ and choose u in the form $u = Th$, where T is the Cauchy-Green transform (2.6). Using Proposition 2.11, we have

$$h = A(u,v,\zeta)(1 + \overline{\zeta\Pi(h)}),$$

where Π is the Ahlfors-Beurling transform from (2.8). By Calderon-Zygmund inequality we can choose $p > 2$ such that $q\,\|\Pi(h)\|_p < 1$. Hence for given $u, v \in L^\infty(\mathbb{D})$ the map $h \to A(1 + \overline{\zeta\Pi(h)})$ is a contraction in $L^p(\mathbb{D})$ and there exists a unique solution $h = h(u,v)$ with

$$\|h\|_{L^p(\mathbb{D})} \leq C\,\|A\|_{L^p(\mathbb{D})}\,,\ C \in \mathbb{R}. \tag{3.6}$$

We consider the map $F\colon L^\infty(\mathbb{D}) \times L^\infty(\mathbb{D}) \to L^\infty(\mathbb{D}) \times L^\infty(\mathbb{D})$ given as

$$F(u,v) = \Big(T(h), T\Big(B(1 + \overline{\zeta\Pi(h)})\Big)\Big) = (U,V).$$

The mapping $T\colon L^p(\mathbb{D}) \to \mathcal{C}^\alpha(\mathbb{C})$, $\alpha = (p-2)/2$, is bounded (Proposition 2.10) and the inclusion $\mathcal{C}^\alpha(\overline{\mathbb{D}}) \subset L^\infty(\mathbb{D})$ is compact. Hence T mapping from $L^p(\mathbb{D})$ to $L^\infty(\mathbb{D})$ is compact. By Schauder Fixed point theorem (Theorem 2.6), it suffices to prove that there exist $u_0, v_0 > 0$ such that the bounded, closed and convex set

$$E = \{(u,v) \in L^\infty(\mathbb{D}) \times L^\infty(\mathbb{D});\, \|u\|_\infty \leq u_0,\ \|v\|_\infty \leq v_0\}$$

is mapped to itself, that is, $F(E) \subset E$.

Since $a(z,0) = b(z,0) = 0$, there exists a constant C' such that $|a(z,w)| < C'\,|w|$ and $|b(z,w)| < C'\,|w|$. Thus we have

$$\|A\|_p \leq C' e^{\|v\|_\infty} n^{-1/p}, \quad \|B\|_p \leq C' e^{\|u\|_\infty}.$$

Using the Proposition 2.10 and (3.6), we may estimate the norm of two components up to some constants $D_1, D_2 > 0$

$$\|U\|_\infty \leq D_1 e^{\|v\|_\infty} n^{-1/p}, \quad \|V\|_\infty \leq D_2 e^{\|u\|_\infty}.$$

We consider the system

$$u = D_1 e^v n^{-1/p}, \qquad v = D_2 e^u.$$

For n big enough the equation

$$u = D_1 e^{D_2 e^u} n^{-1/p}$$

has two roots. We choose u_0 to be the smaller one and set $v_0 = D_2 e^{u_0}$. One now has

$$\|U\|_\infty \leq D_1 e^{v_0} n^{-1/p} \leq u_0, \quad \|V\|_\infty \leq D_2 e^{u_0} \leq v_0.$$

Finally, note that u_0 tends to zero when n tends to ∞. We also have

$$v = T\left(B\left(1+\overline{\zeta\Pi(h)}\right)\right)$$

which implies that the norm of v (and e^v resp.) is bounded in some $\mathcal{C}^\alpha$ space as well. Since $|e^v| = 1$ on $\partial\mathbb{D}$, we have $|w| < \tilde{C}\,|\zeta|^n\,(1-|\zeta|)^\alpha$ for $\zeta \in \mathbb{D}$. Hence $\|w\|_\infty \to 1$ as n tends to ∞. □

3.4. Poletsky theory of discs in almost complex manifolds

In the early 1990's Poletsky [**31**] gave an explicit formula for constructing the largest plurisubharmonic minorant, $\hat{f}$, of an upper semicontinuous function f on a domain in $\mathbb{C}^n$. His formula expresses $\hat{f}$ as the pointwise infimum of averages of f over boundaries of analytic discs with a given center. Poletsky's result, which has found many interesting applications, was subsequently extended to some complex manifolds by Lárusson and Sigurdsson [**24**], to all complex manifolds by Rosay [**32, 34**], and to locally irreducible complex spaces by Drinovec-Drnovšek and Forstnerič [**10, 11**]. In this section, we prove the same result on any smooth 4-dimensional manifold equipped with a non-integrable almost complex structure. The following proposition along with the subsection § 3.3.2. will be the key technical tool.

PROPOSITION 3.11. *Let (M,J) be an almost complex manifold and $f\colon M \to \mathbb{R}\cup\{-\infty\}$ an upper semicontinuous function. Let $u_p\colon \overline{\mathbb{D}} \to M$ be a smooth immersed J-holomorphic disc centered at the point $p \in M$. For every $\epsilon > 0$ there exists a small neighborhood U of p in M so that for every $q \in U$ there is a smooth immersed J-holomorphic disc u_q centered at q with*

$$\int_0^{2\pi} f\circ u_q(e^{it})\frac{\mathrm{d}t}{2\pi} \le \int_0^{2\pi} f\circ u_p(e^{it})\frac{\mathrm{d}t}{2\pi} + \epsilon.$$

The discs u_q depend smoothly on q, and the family $G(\zeta,q) = u_q(\zeta)$, $\zeta \in \overline{\mathbb{D}}$, is locally diffeomorphic.

PROOF. We define Σ and $\tilde{\Upsilon}$ as in the proof of Theorem 3.4. Take a point q' close enough to the origin and define $h_{q'} = \tilde{\Upsilon}^{-1}(q')$, where q' is considered as a constant disc. We obtain discs close to u_p by projecting the map $\Sigma^{-1}(\zeta, h_{q'}(\zeta))$ from $\overline{\mathbb{D}}\times M$ to M. This gives a family of nearby discs whose centers fill a neighborhood of p. The rest can be verified by a straightforward computation. □

An upper semicontinuous function on (M, J) is said to be *J-plurisubharmonic* if its composition with any J-holomorphic disc is subharmonic on the disc. This generalizes naturally the notion of plurisubharmonicity in the integrable case. As in the standard case, the property can be expressed in terms of a certain generalized Levi form, see for instance [**19**]. However, what we need in order to prove the main theorem is solely the basic definition given above.

THEOREM 3.12. *Suppose that (M, J) is a smooth almost complex manifold with $J \in \mathcal{C}^\infty$ and $\dim_{\mathbb{R}} M = 4$. Given an upper semicontinuous function $f\colon M \to \mathbb{R} \cup \{-\infty\}$ and a point $p \in M$, define*

$$\hat{f}(p) = \inf \int_0^{2\pi} f \circ u(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi}.$$

The infimum is over all J-holomorphic discs $u\colon \overline{\mathbb{D}} \to M$ with $u(0) = p$. Then the function $\hat{f}$ is J-plurisubharmonic on M or identically $-\infty$.

Remark 3.13. Every pseudoholomorphic disc can be approximated by a smooth J-holomorphic immersion ([**36**], Theorem 1.1). Thus the infimum can be taken over the immersed discs.

Remark 3.14. It follows from the submeanvalue property of subharmonic functions on the disc $\overline{\mathbb{D}}$ that $\hat{f}$ is the largest J-plurisubharmonic function on M that is bounded above by f.

PROOF. Fix $p \in M$ and $\epsilon > 0$. There exists a smooth immersed J-holomorphic disc u_p centered at p such that

$$\int_0^{2\pi} f \circ u_p(e^{it}) \frac{\mathrm{d}t}{2\pi} < \hat{f}(p) + \epsilon.$$

Take any $q \in M$ close enough to p so that Proposition 3.11 applies. Then

$$\hat{f}(q) \leq \int_0^{2\pi} f \circ u_q(e^{it}) \frac{\mathrm{d}t}{2\pi} \leq \int_0^{2\pi} f \circ u_p(e^{it}) \frac{\mathrm{d}t}{2\pi} + \epsilon < \hat{f}(p) + 2\epsilon.$$

Thus $\hat{f}$ is upper semicontinuous.

Since f is upper semicontinuous, there is a decreasing sequence of continuous functions $f_1 > f_2 > \dots$ such that $f = \lim_{k\to\infty} f_k$ pointwise in M. Clearly, the corresponding functions $\hat{f}_k$ form a decreasing sequence converging to $\hat{f}$. Thus it suffices to prove plurisubharmonicity of $\hat{f}$ in the case when f is continuous.

Finally, we need to show that, given $p \in M$ with $\hat{f}(p) > -\infty$ and for every smooth J-holomorphic immersion u_p centered at p, we have

$$\hat{f}(p) = \hat{f}(u_p(0)) \leq \int_0^{2\pi} \hat{f} \circ u_p(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi}. \tag{3.7}$$

Fix such a disc u_p and a number $\epsilon > 0$. Let $\mathbb{D}_{u_p}$ denote a small neighborhood of $\overline{\mathbb{D}}$ on which u_p is defined, and let $z \in \mathbb{D}_{u_p}$. There exists a smooth immersed J-holomorphic disc v_z centered at $u_p(z)$ such that

$$\int_0^{2\pi} f \circ v_z(e^{it}) \frac{\mathrm{d}t}{2\pi} < \hat{f}(u_p(z)) + \frac{\epsilon}{2\pi}. \tag{3.8}$$

Without loss of generality, we may assume that the disc is transversal to the initial one. We wish to find a family of such close-to-extremal discs, smoothly parametrized by the points in the boundary of u_p. This can be accomplished by using Proposition 3.11; we now explain the construction.

If a point z' is close enough to z, then using Proposition 3.11, one can perturb the disc v_z into a disc $v^z_{z'}$ centered at $u_p(z')$ and in addition satisfying (3.8). Let us cover $\partial\mathbb{D}$ by a finite number of sets

$$U_z = \left\{ z' \in \mathbb{D}_{u_p};\ v^z_{z'} \text{ is well defined} \right\}.$$

Denote them by $U_{z_1}, U_{z_2}, \ldots, U_{z_m}$ and let $U_{z_{m+1}} = U_{z_1}$. We may assume that the triple intersections are empty and that every U_{z_j} meets precisely the sets $U_{z_{j-1}}$ and $U_{z_{j+1}}$. For every $j = 1, \ldots, m$ we pick a point $p_j \in U_{z_j} \cap U_{z_{j+1}}$. On a small neighborhood of $u_p(p_j)$, the set of small J-holomorphic discs is in bijective correspondence with the set of small standard holomorphic discs in $\mathbb{C}^2$ (see proof of the Theorem 3.1). Let V_j be the set of all $z' \in \mathbb{D}_{u_p}$ for which $u_p(z')$ lies in this neighborhood. We now shrink the sets U_{z_j} (keeping the same notation) until they become pairwise disjoint. We may assume that these new sets, together with the sets V_j, still cover $\partial\mathbb{D}$ and that the measure of $E = \partial\mathbb{D} \setminus \cup_{j=1}^m U_{z_j}$ is arbitrarily small.

We define a smooth cut-off function χ_j on every U_{z_j} such that it equals to 1 on a slightly smaller closed subset of $U_{z_j} \setminus (V_j \cup V_{j-1})$ and that it has a small positive value on a slightly smaller closed subset of $U_{z_j} \cap V_j$ and $U_{z_j} \cap V_{j-1}$. We redefine the discs $u^{z_j}_{z'}$ into the maps $\zeta \mapsto u^{z_j}_{z'}(\chi_j(z')\zeta)$, $\zeta \in \overline{\mathbb{D}}$ and obtain a smooth, locally diffeomorphic family defined on $\cup_{j=1}^m U_{z_j} \times \overline{\mathbb{D}}$. The discs with centers in $u(U_{z_j} \cap V_j)$ and $u(U_{z_j} \cap V_{j-1})$ are small. We apply the bijective correspondence defined in the neighborhood of $u_p(p_j)$ to construct discs with centers in $u(V_j \setminus (U_{z_j} \cup U_{z_{j+1}}))$. After translating the problem into $\mathbb{C}^2$, it becomes a simple matter of parametrizing two disjoint families of disc by adding discs with prescribed centers.

Let us denote now by $G(z,\zeta) = u_z(\zeta)$ (we have dropped the upper index) the constructed family defined on $\cup_{j=1}^m (U_{z_j} \cup V_j) \times \overline{\mathbb{D}}$. Since E has small measure and almost every disc satisfies (3.8), we have the following crucial inequality:

$$\int_0^{2\pi}\int_0^{2\pi} f \circ G(e^{i\theta}, e^{it}) \frac{\mathrm{d}t}{2\pi}\frac{\mathrm{d}\theta}{2\pi} < \int_0^{2\pi} f \circ u_p(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi} + \epsilon. \tag{3.9}$$

We now refer to the subsection § 3.3.2. Given an immersed torus $\Lambda = G(\partial\mathbb{D} \times \partial\mathbb{D})$, we construct an admissible parametrization and extend the family G to some neighborhood of the bidisc $\mathbb{D}^2$. Note that a reparametrization of the boundary discs $\zeta \to v_z(e^{i\sigma(\zeta)}\zeta)$ might be needed, but this does not affect the inequalities (3.8) and (3.9).

Further, let $u_{c,N}$ and $v_{c,N}$ be as in Lemma 3.10, defined for $c \in \partial\mathbb{D}$ and $N \in \mathbb{N}$. We choose N big enough so that the disc

$$\varphi_{c,N}(\zeta) = G(\zeta e^{u_{c,N}(\zeta)}, c\zeta^N e^{v_{c,N}(\zeta)})$$

is well defined and satisfies

$$\int_0^{2\pi} f \circ \varphi_{c,N}(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi} < \int_0^{2\pi} f \circ G(e^{i\theta}, e^{i(t+\beta(\theta)+N\theta)}) \frac{\mathrm{d}\theta}{2\pi} + \epsilon.$$

Here $c = e^{it}$ and $v(e^{i\theta}) = e^{i\beta(\theta)}$. We set

$$I_t = \int_0^{2\pi} f \circ G(e^{i\theta}, e^{i(t+\beta(\theta)+N\theta)}) \frac{\mathrm{d}\theta}{2\pi}, \quad t \in [0, 2\pi]$$

By the mean value theorem there exists a value $\nu \in [0, 2\pi)$ such that

$$\begin{aligned} I_\nu &= \int_0^{2\pi}\int_0^{2\pi} f \circ G(e^{i\theta}, e^{i(t+\beta(\theta)+n\theta)}) \frac{\mathrm{d}t}{2\pi}\frac{\mathrm{d}\theta}{2\pi} \\ &= \int_0^{2\pi}\int_0^{2\pi} f \circ G(e^{i\theta}, e^{it}) \frac{\mathrm{d}t}{2\pi}\frac{\mathrm{d}\theta}{2\pi}. \end{aligned}$$

(The second equality follows by a change of variables in the double integral.) This is the key trick due to Poletsky. Together with (3.9) we obtain

$$\hat{f}(u_p(0)) \le \int_0^{2\pi} f \circ \varphi_{e^{i\nu},N}(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi} \le I_\nu + \epsilon \le \int_0^{2\pi} \hat{f} \circ u_p(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi} + 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, (3.7) holds and the theorem is proved. □

Remark 3.15. One of the two crucial steps in the above proof is the construction of a disc attached to a given real torus. In the non-integrable case, such a construction is done only for complex dimension two. In a recent paper [**38**, sec. 5], the result was improved allowing coordinates which are not locally diffeomorphic. Furthermore, an existence of a family of discs whose boundaries fill the whole torus is proved. In higher dimension, the system for u and v becomes overdetermined.

The proof of Rosay [**32**] uses a different approach. First, a non-holomorphic disc with a small $\bar{\partial}$-derivative is attached to the given torus; in the second step, this disc is approximated by a holomorphic disc. Proposition 3.8 above was developed as an attempt to generalize this procedure to the almost complex case. However, we have failed at the first step, namely to find a disc with a small $\bar{\partial}$-derivative. In the standard (integrable) almost complex structure, smallness of the $\bar{\partial}$-derivative is guaranteed by an appropriate construction of the family G. For this to be true, holomorphic dependence in both parameters is needed. Such an assumption seems to be unreachable in general.

We remark also that the method of gluing holomorphic sprays of discs, which was used by Drinovec-Drnovšek and Forstnerič in [**10, 11**], does not apply in the almost complex case.

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Glavni poudarki raziskave v slovenskem jeziku

Doktorska disertacija sodi na področje kompleksne analize. Obravnava mnogoterosti M sode dimenzije, katerih tangentni sveženj TM je opremljen z realnim tenzorjem J z lastnostjo $J^2 = -Id$. Takemu tenzorju pravimo skoraj kompleksna struktura, par (M, J) pa imenujemo skoraj kompleksna mnogoterost.

Kadar imamo na mnogoterosti kompleksno strukturo, t.j. atlas kompatibilnih kart z biholomorfnimi tranzicijskimi preslikavami, je z njo porojena tudi skoraj kompleksna struktura. V okolici vsake točke jo lahko identificiramo kot množenje z imaginarno enoto $i = \sqrt{-1}$. Natančneje, imamo nabor kart $\varphi_\eta \colon U_\eta \to \mathbb{R}^{2n}$, za katere pri pogoju $U_\eta \cap U_{\eta'} \neq \emptyset$ velja, da diferenciali $\mathrm{d}(\varphi'_\eta \circ \varphi_\eta^{-1})$ komutirajo z matriko

$$J_{st} = \begin{bmatrix} 0 & -1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{bmatrix}.$$

Zato je struktura $J := \mathrm{d}\varphi_\eta^{-1} \circ J_{st} \circ \mathrm{d}\varphi_\eta$ dobro definirana na M.

Skoraj kompleksne mnogoterosti so torej naravna razširitev kompleksne kategorije. Razredu skoraj kompleksnih struktur, ki sovpadajo z obstojom kompleksnega atlasa, pravimo integrabilne. Seveda to niso vse strukture. Enega najbolj znanih neintegrabilnih primerov najdemo na sferi $S^6 \subset \mathbb{R}^7$, porojen pa je z množenjem oktonionov (primer 1.9). Ker je vprašanje obstoja kompleksnega atlasa na tej isti mnogoterosti še vedno odprto, je z njim lepo ilustrirano dejstvo, da so analitične omejitve za obstoj kompleksne strukture mnogo strožje od topoloških, ki omejujejo konstrukcijo globalnega tenzorja. To daje skoraj kompleksnim objektom določeno fleksibilnost, preko katere so bili doseženi tudi nekateri za integrabilno teorijo novi rezultati (povzeto po [**25**]).

Poseben primer so mnogoterostih realne dimenzije 2. Na njih so vse skoraj kompleksne strukture integrabilne. Obstoj lokalnih kompleksnih kart se namreč prevede na klasično Beltramijevo enačbo

$$\frac{\partial \varphi}{\partial \bar{\zeta}} = \mu \frac{\partial \varphi}{\partial \zeta} \quad \text{za} \quad \|\mu\|_\infty < 1.$$

Ta je vedno rešljiva na prostoru $L^p(\mathbb{D})$, $p > 2$. V našem primeru je zaradi zadostne regularnosti koeficienta $\mu\colon \mathbb{D} \to \mathbb{C}$ (ta izvira iz regularnosti strukture) rešitev tudi homeomorfna [**1**]. Edine skoraj kompleksne mnogoterosti te dimenzije so torej Riemannove ploskve.

Preslikava $f\colon (M', J') \to (M, J)$ je (J', J)-holomorfna, če njen diferencial zadošča pogoju

$$\mathrm{d}f \circ J' = J \circ \mathrm{d}f \text{ na } TM'.$$

Sistem je v splošnem predoločen, kar pomeni, da ni ne holomorfnih preslikav in ne funkcij. Izjema je primer, ko je $\dim_{\mathbb{R}} M' = 2$. Mnogoterost (M', J') je tedaj lokalno ekvivalentna enotskemu disku $\mathbb{D} \subset \mathbb{R}^2$ opremljenemu s strukturo J_{st}. V tem primeru za f uporabljamo izraz psevdoholomorfna oz. J-holomorfna krivulja.

Začetki študija teh krivulj segajo v 60-ta leta dvajsetega stoletja, ko sta Nijenhuis in Woolf [**28**] dokazala njihov obstoj skozi poljubno točko in v poljubni tangentni smeri (izrek 1.13). Področje se je izjemno razmahnilo v osemdesetih, ko je Gromov [**15**] z njihovo pomočjo dokazal obstoj globalnih simplektičnih invariant in s tem naredil pravo revolucijo na področju simplektične geometrije. Tudi v novejšem času se kažejo številne nove uporabe v topologiji in na drugih področjih, na primer pri definiciji Floerove homologije ali študiju (ne)ekvivalentnosti gladkih struktur. Analiza na skoraj kompleksnih mnogoterostih je tako trenutno eno najaktivnejših področij kompleksne analize in geometrije.

V disertaciji smo se podrobneje posvetili nekompaktnemu primeru, teoriji J-holomorfnih diskov $u\colon \mathbb{D} \to (M, J)$, ki zadoščajo enačbi

$$\mathrm{d}u \circ J_{st} = J \circ \mathrm{d}u.$$

Njihov obstoj omogoča zanimive posplošitve rezultatov iz kompleksne teorije, še posebno tistih, ki so bili dokazani z uporabo analitičnih diskov. Pri tem je ključno, da zgornji, posplošeni Cauchy-Riemannov sistem kljub neintegrabilnosti strukture J ostane eliptičen. To omogoči analogijo s standardnim primerom, na kateri temeljijo naši rezultati.

Lokalni pogled na J-holomorfnost

Denimo, da je preslikava $u : \mathbb{D} \to (\mathbb{R}^{2n}, J)$ psevdoholomorfna in da velja $\det(J + J_{st}) \neq 0$. Pogoj za J-holomorfnost je tedaj mogoče zapisati v ekvivalentni obliki [**33, 38**]:

$$u_{\bar{\zeta}} + A(u)\overline{u_\zeta} = 0, \quad \zeta = x + iy \in \mathbb{D},$$

kjer je

$$A(u) = (J(u) + J_{st})^{-1} (J(u) - J_{st}).$$

Ta zapis je zelo priročen, saj je operator $A(Z)$ kompleksno linearen za vsak $Z \in \mathbb{R}^{2n}$. Zgornjo enačbo lahko tako obravnavamo kot posplošeni Cauchy-Riemannov sistem za preslikave $u\colon \mathbb{D} \to \mathbb{C}^n$. Nadalje velja, da je kompleksna matrika A ničelna natanko tedaj, ko je $J = J_{st}$.

Preko koeficientov matrike a_{jk} je mogoče izraziti tudi ekvivalentne pogoje za integrabilnost strukture J [**36**]:

$$N_{jkl} = N_{jlk}, \quad N_{jkl} := \frac{\partial a_{jk}}{\partial \bar{z}_l} + \sum_{s=1}^{n} a_{sl} \frac{\partial a_{jk}}{\partial z_s}.$$

Potrebnost pogoja izvira iz zahteve po obstoju (J, J_{st})-holomorfne preslikave, zadostnost je posledica Newlander-Nirenbergovega izreka [**27**]. Poudariti velja, da je tak lokalni pogoj ekvivalenten ničelnosti Nijenhuisovega tenzorja - globalnemu pogoju, s katerim je integrabilnost strukture podana v klasičnih referencah [**33**].

Pri obravnavi splošnih mnogoterosti je matrika A odvisna od izbire lokalne karte. Za začetek je potrebno zadostiti pogoju, da je tenzor $J + J_{st}$ v njej obrnljiv. Natančneje, če pri izbrani karti $\varphi\colon U \to \mathbb{R}^{2n}$ na $\varphi(U)$ definiramo strukturo s predpisom

$$\varphi_*(J) := \mathrm{d}\varphi \circ J \circ \mathrm{d}\varphi^{-1},$$

mora zanjo veljati $\det(\varphi_*(J) + J_{st}) \neq 0$. Omejitev ni zelo stroga, saj lahko brez škode za splošnost predpostavimo, da za $p \in U$ velja $\varphi(p) = 0$ in $\varphi_*(J)(0) = J_{st}$ (vsaka matrika J z lastnostjo $J^2 = -Id$ je konjugirana matriki J_{st}).

Identifikacije s standardno strukturo v splošnem ne moremo doseči tudi na okolici točke. To bi že pomenilo tudi izpolnitev integrabilnostnih zahtev. Vseeno lahko, z uporabo raztegov, okolico dane točke skrčimo tako, da na njej dosežemo ujemanje strukture z J_{st} do drugega reda natančno (lema 1.17). To pomeni, da v primerno izbrani karti za matriko A velja $A(0) = 0$, njena $\mathcal{C}^1$-norma pa je poljubno majhna.

Nadaljnjo analizo omogoča transformacijsko pravilo, ki opisuje spremembe matrike A ob difeomorfni zamenjavi koordinat (lema 1.15). Po vzoru Rosaya [**33**] smo primere lokalnih kart, v katerih ima matrika posebno lepe lastnosti, poimenovali normalizacije. Rezultat Diedricha in Sukhova [**9**] pove, da lahko na tak način poleg vrednosti v izhodišču uničimo tudi kompleksne odvode matrike (izrek 1.18). Na drugi strani smo pokazali, da podobna normalizacija v konjugiranem primeru ni možna. Ničelnost antikompleksnih odvodov je namreč povezana z integrabilnostjo strukture in v splošnem nedosegljiva [**33**] (izrek 1.19).

Tudi v primeru vloženega psevdoholomorfnega diska $u\colon \overline{\mathbb{D}} \to M$ lahko na okolici slike $u(\overline{\mathbb{D}})$ najdemo difeomorfizem, ki dani disk izravna

$$\varphi \circ u(\zeta) = u_0(\zeta) := (0, \ldots, 0, \zeta) \in \mathbb{C}^n,$$

strukturo pa standardizira vzdolž slike $u_0(\overline{\mathbb{D}})$ [**19**] (zaključek 1.21). Za kompleksno matriko to pomeni, da je $A(0, \ldots, 0, \zeta) = 0$, $\zeta \in \overline{\mathbb{D}}$. Nadalje lahko s Sukhov-Tumanovo normalizacijo v točkah $u_0(\overline{\mathbb{D}})$ uničimo tudi kompleksne odvode matrike A [**37**] (izrek 1.23). Zavoljo enostavnosti smo slednji rezultat predstavili le v primeru, ko je $n = 2$. Enaka trditev velja tudi v višjerazsežnih kompleksnih prostorih [**33**].

Poglavje smo sklenili z originalnim rezultatom za $\mathbb{C}^2$ [**22**]. Pokazali smo, da na okolici vloženega diska v splošnem ni koordinat, v katerih je struktura $\mathcal{C}^1$-blizu standardni. To pomeni, da nasprotno kot na okolici točke ne moremo predpostaviti da je $\mathcal{C}^1$-norma kompleksne matrike A majhna v lokalni karti, lahko pa jo aproksimiramo z neničelno modelno matriko A_1 (trditev 1.22). Pokazali smo, da je razred modelnih struktur klasificiran s kompleksno funkcijo β, katere modul je biholomorfna invarianta. Natančni rezultati so formulirani v spodnjem izreku.

Izrek. *Naj bo J skoraj kompleksna struktura razreda $\mathcal{C}^3$ definirana na okolici slike $u_0(\overline{\mathbb{D}}) \subset \mathbb{C}^2$ in z lastnostjo $J(u_0(\overline{\mathbb{D}})) = J_{st}$. Naj bosta*

$$B_1(w) = \left(\frac{\partial a_{1,2}}{\partial z}\right)(0, w) \text{ in } B_2(w) = \left(\frac{\partial a_{1,2}}{\partial \bar{z}}\right)(0, w),$$

kjer je $a_{1,2}$ drugi element prve vrstice kompleksne matrike A. Za

$$A_1(z, w) = \begin{bmatrix} 0 & zB_1(w) + \bar{z}B_2(w) \\ 0 & 0 \end{bmatrix}$$

veljajo naslednje trditve:

(1) *Za vsako okolico V slike $u_0(\overline{\mathbb{D}})$ in število $\lambda > 0$ obstaja zamenjava koordinat φ, pri kateri za $\zeta \in \overline{\mathbb{D}}$ in $A' = \mathrm{d}\varphi \circ A \circ \mathrm{d}\varphi^{-1}$ velja: $\varphi(0,\zeta) = (0,\zeta)$, $A'(0,\zeta) = 0$ in $\|A' - A_1\|_{C^1(\overline{V})} \leq \lambda$.*
(2) *Struktura A_1 je lokalno ekvivalentna modelni strukutri*

$$A_\beta(z,w) = \begin{bmatrix} 0 & \bar{z}\beta(w) \\ 0 & 0 \end{bmatrix}.$$

Funkcija β je razreda $\mathcal{C}^1$ in odvisna od B_1 in B_2.
(3) *Modelni strukturi A_β in $A_{\beta'}$ sta ekvivalentni natanko tedaj, ko obstaja neničelna holomorfna funkcija g, da je $\bar{g}\beta' = g\beta$.*
(4) *Vrednost $|\beta|$ je invariantna za spremembe koordinat φ z lastnostma $\varphi(0,\zeta) = (0,\zeta)$ in $\varphi_*(J)(0,\zeta) = J_{st}$ za vsak $\zeta \in \overline{\mathbb{D}}$.*

Lineariziran problem

Velik del rezultatov, ki jih predstavimo v tretjem poglavju, temelji na izreku o implicitni funkciji. V ta namen smo v disertacijo vključili celovito študijo lineariziranega problema, ki bralcu omogoča vpogled v teorijo posplošenih analitičnih vektorjev. Namenili smo ji celotno drugo poglavje, v katerem tako ni originalnih delov. Gre za osnutek preglednega znanstvenega članka.

Posplošeni analitični vektor je vektorska funkcija $u\colon \mathbb{D} \to \mathbb{C}^n$, ki pri danih matričnih funkcijah B_1 in B_2 ustreza enačbi

$$u_{\bar{\zeta}} + B_1 u + B_2 \bar{u} = 0.$$

Obsežno študijo skalarnega primera je opravil Vekua [**39**]. Rešitve enačbe je poimenoval posplošene analitične funkcije. Preko načela podobnosti (Similarity principle) je dano rešitev u izrazil kot $u = \phi e^{\omega}$, kjer sta ϕ holomorfna, ω pa omejena funkcija (izrek 2.12). Opazimo, da se ničelni množici u in ϕ ujemata. To omogoča posplošitev nekaterih klasičnih rezultatov, na primer Liouvillovega izreka.

Vekua je v analizo uvedel tudi Cauchy-Greenov operator

$$T(u)(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{u(\zeta)}{z-\zeta}\,\mathrm{d}x\,\mathrm{d}y(\zeta).$$

Ta na eni strani rešuje klasično $\overline{\partial}$-enačbo, na drugi pa za en red dviguje regularnost v Soboljevem prostoru $W^{k,p}(\mathbb{D})^n$, $p > 2$, ali Hölderjevem prostoru $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})^n$, $0 < \alpha < 1$. V obeh primerih je $k \in \mathbb{N}_0$, indeks n pa pomeni, da gre v splošnem lahko tudi za vektorske funkcije u.

Z uporabo operatorja dobimo ekvivalentno, integralsko obliko osnovne enačbe

$$\Phi(u) := u + T(B_1 u + B_2 \bar{u}) = \phi.$$

V njej smo s ϕ znova označili holomorfno funkcijo. Operator Φ je endomorfizem Banachovih prostorov $W^{k,p}(\mathbb{D})^n$ in $\mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})^n$, ter hkrati kompaktna perturbacija identitete. To porodi Fredholmovo lastnost. V skalarnem primeru $n = 1$ je njegovo jedro trivialno, zato je zveza med posplošenimi analitičnimi in običajnimi holomorfnimi funkcijami, ki je podana z njim, povratno enolična.

Z analizo problema v višjih dimenzijah se je ukvarjalo več avtorjev. Prvi jo je obravnaval Pascali [**30**], zato vektorski obliki enačbe pogosto pravimo Pascalijev sistem. Dokazal je lokalno obliko načela podobnosti: za vsako točko enotskega diska obstaja okolica, na kateri lahko rešitev u izrazimo z $u = S\phi$, pri čemer je vektorska funkcija ϕ je holomorfna, matrična S pa obrnljiva (izrek 2.14). Kljub temu, da je kasneje Buchanan [**3**] dokazal tudi globalno verzijo načela, popolna analogija s skalarnim primerom ni mogoča. Izkaže se namreč, da jedro operatorja Φ ni vedno trivialno. To je posledica odsotnosti Liouvillovega izreka, na katero je opozoril že Habetha [**16**] (primer 2.16). Vseeno vzpostavitev povratno enolične zveze z običajnimi holomorfnimi vektorji ni nemogoča. Konstruirala sta jo Sukhov in Tumanov [**36**], in sicer kot majhno linearno perturbacijo korespondence Φ (izrek 2.18).

Ob zaključku poglavja smo predstavili še nekaj rezultatov, ki se nanašajo na Riemann-Hilbertov problem t.j. robni pogoj oblike

$$\mathrm{Re}(Pu) = \psi \text{ za } P\colon \partial\mathbb{D} \to \mathrm{GL(n,\mathbb{C})} \text{ in } \psi \in \mathcal{C}^{k,\alpha}_{\mathbb{R}}(\partial\mathbb{D})^n.$$

Holomorfna študija tega problema (za $n = 1$) je klasična. Znano je, da je rešljivost robnega problema odvisna od ovojnega števila funkcije P, ki ga označimo z $W(P)$ (npr. [**26**]). Netrivialna holomorfna rešitev obstaja, če in samo če je $W(P) \leq 0$. Natančneje, v tem primeru obstaja $-2W(P)+1$ neodvisnih rešitev za poljuben $\psi \in \mathcal{C}^{k,\alpha}_{\mathbb{R}}(\partial\mathbb{D})^n$. Nasprotno mora funkcija ψ zadostiti $2W(P)-1$ linearno neodvisnim omejitvam, če želimo imeti rešitev za $W(P) > 0$. V obeh primerih je indeks problema, t.j. razlika med številom neodvisnih rešitev homogenega problema in številom neodvisnih omejitev na ψ, enak $-2W(P) + 1$.

V disertaciji smo obravnavali posplošene analoge teh trditev. Skalarni primer je v že omenjeni monografiji obravnaval Vekua [**39**]. Bojarski [**2**], Wendland [**40**], Gilbert in Buchanan [**4, 5**] so problem preučevali tudi v višjih dimenzijah. Po Wendlandovem vzoru smo v študijo vpeljali operator $R = (R_1, R_2)\colon \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})^n \to \mathcal{C}^{k-1,\alpha}(\overline{\mathbb{D}})^n \times \mathcal{C}^{k,\alpha}_{\mathbb{R}}(\partial\mathbb{D})^n$:

$$R_1(v) = v_{\bar\zeta} + B_1 v + B_2 \bar v \text{ in } R_2(v) = \mathrm{Re}\,(Pv)\,.$$

Zanimali sta nas dimenziji $\beta(R) = \dim\ker R$ in $\gamma(R) = \operatorname{codim}\operatorname{range} R$.

Ker je prehod iz holomorfnega problema na primer, ko sta koeficienta B_1 in B_2 neničelna, kompakten, operator R ob njem ostane Fredholmov [**2, 40**]. Njegov indeks je enak:

$$\nu(R) = \beta(R) - \gamma(R) = -2W(\det P) + n.$$

Dovolj je torej eksplicitno izračunati le eno izmed dimenzij.

V skalarno-holomorfnem primeru je ta naloga enostavna. Izračun

$$\beta(R) = \begin{cases} -2W(P) + 1 & \text{if } W(P) \le 0 \\ 0 & \text{if } W(P) > 0 \end{cases},$$

dobimo z običajnim razvojem funkcije na disku. Nadalje zaradi načela podobnosti veljaven tudi, ko sta $B_1, B_2 \neq 0$ (trditev 2.25).

Ko je $n \geq 2$, rezultat ni odvisen od enega samega števila. S pomočjo Birkhoffove faktorizacije matriko $G = -P^{-1}\overline{P}$ razcepimo na produkt $G = \Theta\Lambda\overline{\Theta}^{-1}$, v katerem je funkcija $\Theta\colon \overline{\mathbb{D}} \to \mathrm{GL}(n,\mathbb{C})$ gladka in holomorfna v notranjosti in $\Lambda(\zeta) = \mathrm{diag}\,(\zeta^{\kappa_1}, \zeta^{\kappa_2}, \ldots, \zeta^{\kappa_n})$. Števila $\kappa_j \in \mathbb{Z}$ se imenujejo parcialni indeksi in so enolično določena do vrstnega reda natančno. Njihovo vsoto $\kappa = -2W(\det P)$ imenujemo totalni indeks.

V primeru, ko je $B_1 = B_2 = 0$, velja

$$\beta(R) = \sum_{\kappa_j \geq -1} (\kappa_j + 1) \text{ in } \gamma(R) = \sum_{\kappa_j < -1} (-\kappa_j - 1).$$

Ta račun je za razliko od skalarnega primera pomanjkljiv, ko je $n \geq 2$ in sta koeficienta B_1 in B_2 neničelna. Ob uporabi načela podobnosti in prehodu na holomorfni problem se namreč ohrani le totalni indeks, ne pa tudi parcialni indeksi (primer 2.28). To pomeni, da razlika obeh dimenzij, t.j. indeks $\nu(R)$, ostane enaka, vendar pa je pri izračunu posamezne dimenzije potrebno upoštevati tudi celoštevilski defekt, ki opiše zvezo med matrikami B_1, B_2 in P. S pomočjo teorije integralskih operatorjev ga je eksplicitno podal Buchanan [**4**] (zaključek 2.29).

Deformacije J-holomorfnih diskov

V drugem poglavju smo v disertacijo vpeljali Cauchy-Greenov operator T. Lokalni pogoj za J-holomorfnost lahko zato izrazimo v ekvivalentni, integralski obliki:

$$\Phi^J(u) := u + T(A(u)\overline{u_\zeta}) = \phi, \quad \zeta = x + iy \in \mathbb{D}.$$

Tako dobimo korespondenco med J-holomorfnimi in običajnimi analitičnimi diski ϕ iste regularnosti v $\mathbb{C}^n$, kjer je $\dim_\mathbb{R} M = 2n$.

Omenili smo, da je na okolici dane točke mogoče najti lokalne koordinate, v katerih je struktura $\mathcal{C}^1$- blizu standardni. Natančneje, obstaja karta, v kateri je $\mathcal{C}^1$-norma kompleksne matrike A dovolj majhna, da je operator Φ^J obrnljiv. To dejstvo uporabimo za dokaz klasičnega Nijenhuis-Woolfov izreka: skozi poljubno točko in v poljubni tangentni smeri obstaja (majhen) J-holomorfen disk, ki je gladko odvisen od začetnih podatkov in strukture J [**35**]. Ob zadostni regularnosti strukture je interpolacija možna tudi za višji red odvodov [**19**] (izrek 3.1).

Narava problema se spremeni, ko operator opazujemo na okolici (netočkastega) J-holomorfnega diska. Povedali smo, da četudi izberemo karto, v kateri je dani disk raven in je struktura standardna vzdolž slike $u_0(\overline{\mathbb{D}})$, operator Φ^J ni nujno majhna perturbacija identitete. Vseeno poznavanje linearne teorije omogoča nadaljno analizo.

Odvod operatorja Φ^J vzdolž u_0 je v tej karti ekvivalenten integralski obliki Pascalijevega sistema, koeficienta B_1 in B_2 pa sta porojena iz odvodov zadnjega stolpca kompleksne matrike A. V posebnem primeru, če je $n = 2$ in je disk vložen, enodimenzionalni normalni prostor ustreza teoriji posplošenih analitičnih funkcij. Tako je za modelno strukturo oz. kompleksno matriko A_1 linearizirani operator obrnljiv. Posledično dobimo obstoj bijektivne korespondence med bližnjimi J-holomorfnimi diski in holomorfnimi diski v $\mathbb{C}^2$ [**22**] (trditev 3.3).

Zgornja, originalna trditev je bila nedavno posplošena tudi za višjerazsežne mnogoterosti. V disertacijo smo zato vključili rezultat Sukhova in Tumanova [**36**]. Ker za $n > 2$ jedro lineariziranega integralskega operatorja ni nujno trivialno, avtorja to odpravita z majhno korekcijo korespondence Φ^J. Ker pristop ni dimenzijsko omejen, lahko obravnavamo okolico poljubnega J-holomorfnega diska - njegov graf je vložen v mnogoterost $(M \times \mathbb{R}^{2n}, J \oplus J_{st})$. Množica J-holomorfnih diskov je tako mnogoterost generirana nad holomorfnimi diski v prostoru $\mathbb{C}^n$.

Podobno analiziramo tudi okolico diskov pripetih na totalno realne podmnogoterosti $E \subset M$ dimenzije $\dim_{\mathbb{R}} E = n$. Njihov J-kompleksni podprostor tangentnega prostora T_pE je trivialen za vsak $p \in E$ t.j. $T_pE \cap J(p)T_pE = \{0\}$. Lokalno, na primer na okolici danega diska, lahko tako mnogoterost podamo kot ničelno množico gladke preslikave $\rho = (\rho_1, \rho_2, \ldots, \rho_n)\colon M \to \mathbb{R}^n$ z lastnostjo $\partial\rho_1 \wedge \partial\rho_2 \wedge \ldots \wedge \partial\rho_n \neq 0$. Nadalje za disk u pravimo, da je pripet na E, če zanj velja $\rho \circ u = 0$ na $\partial\mathbb{D}$. Njemu bližnje, pripete diske zato opiše množica

$$\mathcal{M} = \{v \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})^n \colon u \text{ blizu } v,\ R^J(u) = 0\},$$

kjer je $R^J = (R_1^J, R_2^J)\colon \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})^{n+1} \to \mathcal{C}^{k-1,\alpha}(\overline{\mathbb{D}})^{n+1} \times \mathcal{C}_{\mathbb{R}}^{k,\alpha}(\partial\mathbb{D})^{n+1}$

$$R_1^J(u) = u_{\bar{\zeta}} + A(u)\overline{u_\zeta} \ \text{ na } \overline{\mathbb{D}}, \quad R_2^J(u) = \rho \circ u \ \text{ na } \partial\mathbb{D}.$$

Zanimali so nas zadostni pogoji za to, da je $\mathcal{M}$ mnogoterost.

Problem znova normaliziramo na okolico ravnega diska u_0 in strukturo, ki je standardna vzdolž slike $u_0(\overline{\mathbb{D}})$. V linearizirani obliki dobimo Pascalijev sistem z Riemann-Hilbertovim robnim pogojem $\mathrm{Re}(Pu)$. Kot elementi matrike P v njem nastopajo kompleksni odvodi funkcije ρ. Ker je problem v tangentni smeri holomorfen, je dovolj obravnavati normalno smer. Zadostne pogoje smo tako izrazili v jeziku Wendlandovega operatorja R dimenzije $n-1$. Po izreku o implicitni funkciji je množica $\mathcal{M}$ mnogoterost, če je R surjektiven oziroma, ko je $\gamma(R) = 0$.

V primeru, ko je $n = 2$, je dovolj, da je ovojno število $W(P) \geq 0$ oz. da je totalni indeks $\kappa \geq 0$. Podoben rezultat so dokazali Forstnerič [**12**] v integrabilnem primeru, ter Lizan, Hofer in Sikorav v splošnem [**18**]. Obakrat je bil pogoj izražen v obliki pozitivnosti Maslovega indeksa.

Ko je $n \geq 3$, je rezultat odvisen od $n-1$ indeksov. V standardnem primeru zadostuje, če so ti večji ali enaki od -1, kar je dokazal že Globevnik [**14**]. Originalni zaključek pove, da to v splošnem ni dovolj. Iz linearne teorije je razvidno, da je dimenzija $\gamma(R)$ odvisna tudi od strukture [**21**]. Konkretno, v zadostnem pogoju se pojavita dimenzija jedra Buchananovega integralskega operatorja N in defekt r, ki opisuje zvezo med strukturo in robnimi pogoji [**4**].

Zaključek. *Množica $\mathcal{M}$ je mnogoterost, če velja:*

$$\sum_{k_j < -1}^{n} (-\kappa_j - 1) + N = r.$$

Opomba: $N = r = 0$ v integrabilnem primeru in če je $n = 2$.

Poletskyjeva teorija diskov

V zgodnjih devetdesetih je Poletsky [**31**] konstruiral plurisubharmonične funkcije v prostoru $\mathbb{C}^n$, in sicer kot ogrinjače funkcionalov definiranih na analitičnih diskih. Njegov rezultat je bil nato večkrat posplošen. Najprej za nekatere posebne primere kompleksnih mnogoterosti (Lárusson in Sigurdsson [**24**]), kasneje pa tudi za splošen integrabilni primer (Rosay [**32, 34**]). Nedavno je bila podobna trditev dokazana tudi v lokalno ireducibilnih kompleksnih prostorih (Forstnerič in Drinovec-Drnovšek [**10, 11**]). V disertaciji predstavljamo novo posplošitev na skoraj kompleksne mnogoterosti realne dimenzije štiri [**23**].

Izrek. *Denimo, da je M skoraj kompleksna mnogoterost opremljena z gladko strukturo J in $\dim_{\mathbb{R}} M = 4$. Za navzgor polzvezno funkcijo $f\colon M \to \mathbb{R} \cup \{-\infty\}$ in točko $p \in M$ definiramo*

$$\hat{f}(p) = \inf \int_0^{2\pi} f \circ u(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi}.$$

Infimum je vzet po vseh J-holomorfnih diskih $u\colon \overline{\mathbb{D}} \to M$ z $u(0) = p$. Funkcija $\hat{f}$ je J-plurisubharmonična ali identično enaka $-\infty$.

Opomba: Vsak J-holomorfni disk se da aproksimirati z J-holomorfno imerzijo [**36**], zato lahko infimum vzamemo po imerziranih diskih.

Podobno kot v integrabilnem primeru je funkcija na skoraj kompleksni mnogoterosti J-plurisubharmonična, če je navzgor polzvezna in je njena kompozicija s poljubnim J-holomorfnim diskom subharmonična. V nadaljevanju povzemamo dokaz obeh lastnosti, v katerem smo uporabili razvito deformacijsko teorijo.

Naj bo u_p J-holomorfni disk z lastnostjo $u_p(0) = p \in M$. Zanj rečemo, da je za dani $\epsilon > 0$ blizu ekstremne vrednosti, če velja

$$\int_0^{2\pi} f \circ u_p(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi} < \hat{f}(p) + \epsilon.$$

Ker ga lahko gladko in holomorfno deformiramo, obstaja okolica U točke p, da za vsak $q \in U$ obstaja J-holomorfni disk u_q z lastnostjo $u_q(0) = q$, za katerega je razlika med $f \circ u_q(e^{i\theta})$ in $f \circ u_p(e^{i\theta})$ majhna za vsak $\theta \in [0, 2\pi)$. Še več, predpostavimo lahko, da je u_q blizu ekstremni vrednosti za isti $\epsilon > 0$. To porodi navzgor polzveznost funkcije $\hat{f}$:

$$\hat{f}(q) \le \int_0^{2\pi} f \circ u_q(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi} \le \int_0^{2\pi} f \circ u_p(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi} + \epsilon \le \hat{f}(p) + 2\epsilon.$$

Nadalje želimo dokazati, da je kompozicija $f \circ u_p$ subharmonična za vsak J-holomorfni disk u_p. Dovolj je videti, da velja

$$\hat{f}(u_p(0)) \leq \int_0^{2\pi} \hat{f} \circ u_p(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi}.$$

Naj bo $z \in \partial\mathbb{D}$. Za $\frac{\epsilon}{2\pi} > 0$ izberemo J-holomorfni disk v_z s centrom v točki $u_p(z)$, ki je blizu ekstremni vrednosti. Tako definiramo družino $G(z, \zeta) := v_z(\zeta)$. Z uporabo deformacijske teorije lahko zagotovimo, da je odvisnost od parametra z na okolici roba $\partial\mathbb{D}$ odsekoma gladka oz. gladka, če blizu nezveznosti diske v_z zamenjamo z ustreznimi majhnimi iz Nijenhuis-Woolfovega izreka. Tako dobimo družino z lastnostmi: $G(z, 0) = u_p(z)$, diski $w \mapsto G(z, w)$, $z \in \overline{\mathbb{D}}$, so J-holomorfni in

$$\int_0^{2\pi} \int_0^{2\pi} \hat{f} \circ G(e^{i\theta}, e^{it}) \frac{\mathrm{d}\theta}{2\pi} \frac{\mathrm{d}t}{2\pi} \leq \int_0^{2\pi} \hat{f} \circ u_p(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi} + \epsilon.$$

Nadalje potrebujemo J-holomorfen disk φ centriran v $\varphi(0) = u_p(0)$ in z robnimi vrednostmi v $\Lambda = G(\partial\mathbb{D} \times \partial\mathbb{D})$. V neintegrabilnem primeru je njegova konstrukcija znana le za primer mnogoterosti realne dimenzije štiri [**38**]. Postopek je sledeč. S konstrukcijo majhnih diskov, transverzalnih na točke $u_p(z)$, družino G dopolnimo do lokalnega difeomorfizma na okolici bidiska $\overline{\mathbb{D}}^2$ (pri tem je morda potrebna reparametrizacija roba, a to ne vpliva na zgornjo ekstremalno lastnost družine). Zaradi posebnih lastnosti difeomorfizma G ima kompleksna matrika strukture $J' = \mathrm{d}G \circ J \circ \mathrm{d}G^{-1}$ naslednji dve lastnosti: njen drugi stolpec je ničeln, v horizontalni smeri pa J'-holomorfne diske opisuje Beltramijeva enačba. Za poljubna $n \in \mathbb{N}$ in $c \in \partial\mathbb{D}$ tako obstaja rešitev našega robnega problema, ki je oblike

$$\varphi(\zeta) = G(\zeta e^{u(\zeta)}, c\zeta^n e^{v(\zeta)}), \quad \mathrm{Re}(u) = \mathrm{Re}(v) = 0 \text{ na } \partial\mathbb{D}.$$

To dokažemo s Shauderjevim izrekom (lema 3.10).

V Rosayevem dokazu za integrabilni primer [**32**], ki mu sledimo sicer, je pristop nekoliko drugačen. Na torus Λ je najprej pripet disk $\tilde{\varphi}(\zeta) = G(\zeta, c\zeta^n)$. Ta sicer ni holomorfen, a je primerno izbrana norma njegovega $\bar{\partial}$-operatorja majhna, če je $n \in \mathbb{N}$ velik. Zato je možna njegova aproksimacija s holomorfnim diskom. Dobljeni disk je tako le približno pripet na torus Λ, a to vseeno zadošča potrebam dokaza. Prednost njegove konstrukcije je, da za razliko od zgornje ni odvisna od dimenzije.

V želji po posplošitvi zgornjega pristopa smo razvili metodo, ki omogoča podobno aproksimacijo z J-holomorfnimi diski v prostorih $\mathbb{R}^{2n}$ (trditev 3.8). Te kasneje nismo uporabili, saj se je izkazalo, da je majhnost $\overline{\partial}$-odvoda diska $\tilde{\varphi}$ v integrabilnem primeru povezana s holomorfno odvisnostjo družine $G(z, \zeta)$ od parametra z. Ta je v splošnem nedosegljiva. Tudi sicer smo se omejili le na evklidske prostore, saj se pri obravnavi tega problema na mnogoterostih pojavi nekaj težav že v integrabilnem primeru. Omenjeni aproksimacijski rezultat je tako le delen, a vseeno originalen [**23**].

Kakorkoli, dokaz glavnega izreka sklenemo z nizom neenakosti:

$$\hat{f}(u_p(0)) \leq \int_0^{2\pi} f \circ \varphi(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi} \leq \int_0^{2\pi} \int_0^{2\pi} f \circ G(e^{i\theta}, e^{it}) \frac{\mathrm{d}t}{2\pi} \frac{\mathrm{d}\theta}{2\pi} + \epsilon.$$

Prvi del sledi iz definicije. Drugega dobimo v dveh korakih. Najprej uporabimo lastnost funkcije u (iz definicije diska φ), ki je zgoraj nismo navedli - njena neskončna norma je za velike $n \in \mathbb{N}$ poljubno majhna. Nato sledi integralski trik, ki ga je v svojem članku uporabil tudi Poletsky. Po izreku o povprečni vrednosti lahko parameter $c \in \partial\mathbb{D}$ nadomestimo z uteženim integralom po spremenljivki t in nato s substitucijo dobimo zgornji dvojni integral. Ker so bili robni diski družine G blizu ekstremnim vrednostim, nadalje velja

$$\hat{f}(p) \leq \int_0^{2\pi} \int_0^{2\pi} f \circ G(e^{i\theta}, e^{it}) \frac{\mathrm{d}t}{2\pi} \frac{\mathrm{d}\theta}{2\pi} + \epsilon \leq \int_0^{2\pi} \hat{f} \circ u_p(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi} + 2\epsilon.$$

Dokaz je tako zaključen, saj je bila konstanta $\epsilon > 0$ poljubna.

Izjava o avtorstvu in dovoljenje za objavo

Podpisani Uroš Kuzman izjavljam, da sem doktorsko disertacijo z naslovom Mnogoterosti J-holomorfnih diskov izdelal samostojno pod mentorstvom prof. dr. Franca Forstneriča in somentorstvom prof. dr. Barbare Drinovec-Drnovšek in da Fakulteti za matematiko in fiziko Univerze v Ljubljani dovoljujem objavo elektronske oblike svojega dela na spletnih straneh.

Ljubljana, 12. 3. 2013